Contents

.

Lis	st of F	igures	xiii
List of Tables		xvii	
Pre	eface		xix
Acknowledgments		xxxi	
Int	roduc	tion	xxxv
Ab	raham	A. Ungar	
1.	THO	MAS PRECESSION: THE MISSING LINK	1
	1	A Brief History of the Thomas Precession	1
	2	The Einstein Velocity Addition	3
	3	Thomas Precession and Gyrogroups	6
	4	The Relativistic Composite Velocity Reciprocity Principle	8
	5	From Thomas Precession to Thomas Gyration	11
	6	Solving Equations in Einstein's Addition, and the Einstein Coaddition	13
	7	The Abstract Einstein Addition	16
	8	Verifying Algebraic Identities of Einstein's Addition	18
	9	Matrix Representation of the Thomas Precession	24
	10	Graphical Presentation of the Thomas Precession	27
	11	The Thomas Rotation Angle	29
	12	The Circular Functions of the Thomas Rotation Angle	31
	13	Exercises	34
2.	GYR	OGROUPS: MODELED ON EINSTEIN'S ADDITION	35
	1	Definition of a Gyrogroup	36
	2	Examples of Gyrogroups	39
	3	First Theorems of Gyrogroup Theory	43
	4	Solving Gyrogroup Equations	47
	5	The Gyrosemidirect Product Group	49
	6	Understanding Gyrogroups by Gyrosemidirect Product Groups	52
	7	Some Basic Gyrogroup Identities	57

8	Exercises	71
3. T	HE EINSTEIN GYROVECTOR SPACE	73
1	Einstein Scalar Multiplication	73
2	Einstein's Half	76
3	Einstein's Metric	77
4	Metric Geometry of Einstein Gyrovector Spaces	80
5	The Einstein Geodesics	84
6	Gyrovector Spaces	86
7	Solving a Simple System of Two Equations in a Gyrovector Space	89
8	Einstein's Addition and The Beltrami Model of Hyperbolic Geometry	90
9	The Riemannian Line Element of Einstein's Metric	93
1	0 Exercises	94
4. H	YPERBOLIC GEOMETRY OF GYROVECTOR SPACES	95
1	Rooted Gyrovectors	95
2	Equivalence Classes of Gyrovectors	98
3	The Hyperbolic Angle	104
4	Hyperbolic Trigonometry in Einstein's Gyrovector Spaces	107
5	From Pythagoras to Einstein: The Hyperbolic Pythagorean	
	Theorem	110
6	The Relativistic Dual Uniform Accelerations	112
7	Einstein's Dual Geodesics	114
8	The Riemannian Line Element of Einstein's Cometric	119
9	Moving Cogyrovectors in Einstein Gyrovector Spaces	122
1	0 Einstein's Hyperbolic Coangles	123
1	1 The Gyrogroup Duality Symmetry	126
1:	2 Parallelism in Cohyperbolic Geometry	127
1	3 Duality, And The Dual Gyrotransitive Laws of Mutually Dual Geodesics	128
14	4 The Bifurcation Approach to Hyperbolic Geometry	130
1.	5 The Gyroparallelogram Addition Rule	132
1	6 Gyroterminology	137
1'	7 Exercises	139
5. TI	HE UNGAR GYROVECTOR SPACE	141
1	The Ungar Gyrovector Space of Relativistic Proper Velocities	141
2	Some Identities for Ungar's Addition	145
3	The Gyrovector Space Isomorphism Between Einstein's and Ungar's Gyrovector Spaces	1/6
4	The Riemannian Line Flements of The Unger Dual Matrice	140
5	The Ungar Model of Hyperbolic Geometry	152
5	- the submitted of higher bolic Ocollicity	155

Contents	ix

	6 7	Angles in The Ungar Model of Hyperbolic Geometry The Angle Measure in Einstein's and in Ungar's Curousster	154
	/ 0	Spaces	156
	ð	of Hyperbolic Geometry	158
	9	Exercises	160
6.	THE	MÖBIUS GYROVECTOR SPACE	161
	1	The Gyrovector Space Isomorphism	161
	2	Möbius Gyrovector Spaces	163
	3	Gyrotranslations – Left and Right	168
	4	The Hyperbolic Pythagorean Theorem in the Poincaré Disc Model of Hyperbolic Geometry	170
	5	Gyrolines and the Cancellation Laws	174
	6	The Riemannian Line Elements of the Möbius Dual Metrics	176
	7	Rudiments of Riemannian Geometry	183
	8	The Möbius Geodesics and Angles	184
	9	Hyperbolic Trigonometry in Möbius Gyrovector Spaces	186
	10	Numerical Demonstration	193
	11	The Equilateral Gyrotriangle	201
	12	Exercises	210
7.	GYR	OGEOMETRY	211
	1	The Möbius Gyroparallelogram	211
	2	The Triangle Angular Defect in Gyrovector Spaces	213
	3	Parallel Transport Along Geodesics in Gyrovector Spaces	216
	4	The Triangular Angular Defect And Gyrophase Shift	222
	5	Polygonal And Circular Gyrophase Shift	224
	6	Gyrovector Translation in Möbius Gyrovector Spaces	226
	7	Triangular Gyrovector Translation of Rooted Gyrovectors	232
	8	The Hyperbolic Angle and Gyrovector Translation	234
	9	Triangular Parallel Translation of Rooted Gyrovectors	236
	10	The Nonclosed Circular Path Angular Defect	240
	11	Gyroderivative: The Hyperbolic Derivative	245
	12	Parallelism in Cohyperbolic Geometry	249
	13	Exercises	252
8.	GYR	OOPERATIONS – THE $SL(2, C)$ APPROACH	253
	1	The Algebra Of The $SL(2, C)$ Group	253
	2	The $SL(2, C)$ General Vector Addition	259
	3	Case I – The Einstein Gyrovector Spaces	264
	4	Case II – The Möbius Gyrovector Spaces	266
	5	Case III – The Ungar Gyrovector Spaces	269
	6	Case IV – The Chen Gyrovector Spaces	272

7	Gyrovector Space Isomorphisms	275
8	Conclusion	277
9	Exercises	277
9. THE	COCYCLE FORM	279
1	The Real Einstein Gyrogroup and its Cocycle Form	279
2	The Complex Einstein Gyrogroup and its Cocycle Form	281
3	The Möbius Gyrogroup and its Cocycle Form	283
4	The Ungar Gyrogroup and its Cocycle Form	284
5	Abstract Gyrocommutative Gyrogroups with Cocycle Forms	285
6	Cocycle Forms, By Examples	287
7	Basic Properties of Cocycle Forms	290
8	Applications of the Real Even Cocycle Form Representation	293
9	The Secondary Gyration of a Gyrocommutative Gyrogroup with a Complex Cocycle Form	294
10	The Gyrogroup Extension of a Gyrogroup with a Cocycle Form	295
11	Cocyclic Gyrocommutative Gyrogroups	304
12	Applications of Gyrogroups to Cocycle Forms	309
13	Gyrocommutative Gyrogroup Extension by Cocyclic Maps	310
14	Exercises	311
10.THE	LORENTZ GROUP AND ITS ABSTRACTION	313
1	Inner Product and the Abstract Lorentz Boost	314
2	Extended Automorphisms of Extended Gyrogroups	316
3	The Lorentz Boost of Relativity Theory	321
4	The Parametrized Lorentz Group and its Composition Law	323
5	The Parametrized Lorentz Group of Special Relativity	325
11.THE	LORENTZ TRANSFORMATION LINK	329
1	Group Action on Sets	330
2	The Galilei Transformation of Structured Spacetime Points	332
3	The Galilean Link	335
4	The Galilean Link By a Rotation	335
5	The Lorentz Transformation of Structured Spacetime Points	338
6	The Lorentz Link By a Rotation	343
7	The Lorentz Boost Link	347
8	The Little Lorentz Groups	348
9	The Relativistic Shape of Moving Objects	349
10	The Shape of Moving Circles	352
11	The Shape of Moving Spheres	354
12	The Shape of Moving Straight Lines	358
13	The Shape of Moving Curves	359
14	The Shape of Moving Harmonic Waves	360
15	The Relativistic Doppler Shift	362

411

16 17	Simultaneity: Is Length Contraction Real?	367
17	Come Back	369
18	Exercises	370
12. O TH	IER LORENTZ GROUPS	371
1	The Proper Velocity Ungar-Lorentz Boost	371
2	The Proper Velocity Ungar-Lorentz Transformation Group	373
3	The Unique Ungar-Lorentz Boost that Links Two Points	374
4	The Möbius–Lorentz Boost	375
5	The Unique Möbius-Lorentz Boost that Links Two Points	376
6	The Möbius–Lorentz Transformation Group	377
13.REF	ERENCES	381
About th	ne Author	403
Topic Index		405

Author Index