Contents

1 Introduction **1**

1.1 Fusion . . . 1

1.1.1 Magnetic confinement . . . 2

1.1.2 Magnetic equilibrium . . . 3

1.1.3 Classical and neo-classical transport . . . 3

1.1.4 Turbulent transport . . . 4

1.2 Measuring transport . . . 5

1.2.1 Steady-state and perturbative transport analysis . . . 5

1.2.2 Overview of perturbative experiments . . . 6

1.2.3 Perturbative experiments: actuator and sensor . . . 7

1.3 Methods for estimating transport coefficients . . . 9

1.4 Objectives and contributions of this thesis . . . 11

1.4.1 Activities and layout . . . 11

1.4.2 Specific contributions . . . 12

1.5 Outline of this thesis . . . 14

2 Mathematical modeling of heat transport in tokamaks and stellarators **17**

2.1 Transport modeling . . . 18

2.1.1 Conservation of energy and particles . . . 18

2.1.2 Perturbative transport analysis . . . 19

2.1.3 Slab geometry representation and its relationship to cylindrical geometry . . . 20

2.2 Analytical solutions in the Laplace domain . . . 21

2.2.1 Slab geometry . . . 21

2.2.2 Cylindrical geometry . . . 22

2.3 Logarithmic derivative and transfer function . . . 23

2.3.1 Semi-infinite domain and logarithmic temperature derivative . . . 23

2.3.2 Transfer function and spatial derivatives 26
2.3.3 Double spatial derivatives of A and ϕ 28
2.4 Directionality of heat waves 29
2.5 Conclusion and summary 31

3 Estimation of the transport coefficients using slab geometry (semi-infinite) 33
3.1 Derivation of explicit approximations based on slab geometry . . 34
3.1.1 Approximations for χ in the presence V and τ_{inv} 34
3.1.2 Approximations for V and τ_{inv} 35
3.1.3 Approximations assuming $\tau_{inv} = 0$ 36
3.2 Estimating χ under influence of V and τ_{inv} 37
3.2.1 Diffusivity only 37
3.2.2 Diffusivity and damping 38
3.2.3 Diffusivity, convectivity, and damping 39
3.3 Estimating the convectivity and damping 41
3.3.1 Estimation of V and τ_{inv} in a semi-infinite cylindrical geometry 41
3.3.2 The effect of boundary conditions and radial dependent profiles 43
3.4 Summary 45

4 Estimation of the transport coefficients for heat waves propagating outwards (semi-infinite) 49
4.1 Derivation of explicit approximations 50
4.1.1 Continued fractions 50
4.1.2 Asymptotic expansions 52
4.1.3 Multiple harmonics 55
4.2 Outward solutions 56
4.2.1 Overview of possible explicit approximations 57
4.2.2 Diffusivity only 57
4.2.3 Diffusivity and damping only 61
4.2.4 Diffusivity and convectivity with $\tau_{inv} = 0$ and $\tau_{inv} = 2$ . . 62
4.2.5 Summary 62
4.3 Choice and validation of approximations 63
4.4 Conclusions and summary 65

5 Estimation of the transport coefficients for heat waves propagating inwards (symmetry) 67
5.1 Derivation of explicit approximations using continued fractions . 67
5.1.1 Derivation of inward approximations for the diffusivity and the damping only 68
5.1.2 Diffusivity, convectivity, and damping 69
5.2 Inward solutions 71

5.2.1 Overview of possible explicit approximations 71
5.2.2 Selection of interesting approximations 75
5.2.3 Diffusivity and damping only 76
5.2.4 Diffusivity and convectivity with $\tau_{inv} = 0$ and $\tau_{inv} = 2$ . . 76
5.3 Conclusion and summary 77

6 Estimation of the diffusivity taking frequency measurement uncertainties into account 81
6.1 Distributions of phase and amplitude and its spatial derivatives . 83
6.1.1 Gaussian noise as the result of the central limit theorem . 83
6.1.2 Normal complex circular distributed noise 84
6.1.3 Amplitude and phase distributions and their confidence bounds 85
6.1.4 Distributions of ϕ' and A'/A 87
6.2 Distributions of the diffusivity χ 90
6.2.1 Inverse non-central chi-squared distribution 90
6.2.2 Confidence bounds non-central inverse chi-squared distribution 91
6.2.3 Inverse product distribution function 92
6.3 Estimating means and (co-)variances from measurements 94
6.3.1 Noise distribution of ASDEX Upgrade measurements . . . 94
6.3.2 Estimating the Fourier coefficients and variances 95
6.3.3 Resulting A'/A and ϕ' for AUG 17175 at $\rho_t = 0.473$ and $\rho_t = 0.484$ 98
6.4 Estimating χ 100
6.4.1 Combining amplitude and phase estimates 100
6.4.2 Combining different harmonics for ϕ' and A'/A only . . . 103
6.4.3 Combining different harmonics using the product $\phi' A'/A$ 106
6.4.4 Calibration errors 110
6.4.5 Summary estimating χ with confidence 111
6.5 Conclusions and discussion 113

7 Frequency domain sample maximum likelihood estimation for spatially dependent parameter estimation in PDEs 115
7.1 Introduction 115
7.2 Modeling 118
7.2.1 Considered partial differential equation 119
7.2.2 Local domain based on two measurements 119
7.2.3 Local domain based on three measurements 120
7.2.4 Change of variables 120
7.3 Sample maximum likelihood estimator 121
7.3.1 Error model: errors-in-variables 122
7.3.2 Maximum likelihood cost 122

7.3.3 Optimization and confidence bounds 123
7.3.4 Cost function model validation 124
7.3.5 Input design and choice of domain 124
7.4 Simulation results 125
7.4.1 Estimator and confidence bound validation 126
7.4.2 Finite difference simulation 126
7.4.3 Model validation of finite difference simulation 127
7.5 Conclusions and discussion 129

8 Conclusions, discussion and recommendations 131
8.1 Conclusions 131
8.2 Discussion and recommendations 135
8.2.1 Improvement of the estimation 135
8.2.2 Study of transport 138
8.2.3 Extension to other fields 139

A Analytic eigenfunctions of PDEs 141
A.1 The Bessel and confluent hypergeometric ODEs 141
A.2 Power series representation 142
A.2.1 Example: derivation of power series solution of the Bessel function of first kind of order $\nu = 0$ 143
A.3 Numerical evaluation 145
A.4 Solutions in terms of Bessel and confluent hypergeometric functions 147
A.4.1 Cylindrical geometry with constant coefficients 147

B Derivation of approximations using continued fractions 151
B.1 Continued J-fraction of the ratio of Bessel functions of the second kind 151
B.2 Continued C-fraction of the ratio of Bessel functions of the second kind 152
B.3 Continued T-fraction of the ratio of Bessel functions of the first kind 152
B.4 Continued C-fraction of confluent hypergeometric functions . . . 153
B.5 Continued J-fraction in case $V = \tau_{inv} = 0$ 156
B.6 Approximation for the continued S-fraction for $V = \tau_{inv} = 0$ . . 156
B.7 Asymptotic expansion based on the Bessel function of the second kind 157

C Distribution functions of A and ϕ and numerical calculation of confidence bounds 159
C.1 Distributions of amplitude and phase 159
C.2 Numerical calculation confidence bounds 160

Summary 175
Societal summary 179
Acknowledgements 181
List of publications 185
Curriculum vitae 189