

Contents

<i>A book's apology</i>	xviii
<i>Index of notation</i>	xxii
1 Reminders: convergence of sequences and series	1
1.1 The problem of limits in physics	1
1.1.a Two paradoxes involving kinetic energy	1
1.1.b Romeo, Juliet, and viscous fluids	5
1.1.c Potential wall in quantum mechanics	7
1.1.d Semi-infinite filter behaving as waveguide	9
1.2 Sequences	12
1.2.a Sequences in a normed vector space	12
1.2.b Cauchy sequences	13
1.2.c The fixed point theorem	15
1.2.d Double sequences	16
1.2.e Sequential definition of the limit of a function	17
1.2.f Sequences of functions	18
1.3 Series	23
1.3.a Series in a normed vector space	23
1.3.b Doubly infinite series	24
1.3.c Convergence of a double series	25
1.3.d Conditionally convergent series, absolutely convergent series	26
1.3.e Series of functions	29
1.4 Power series, analytic functions	30
1.4.a Taylor formulas	31
1.4.b Some numerical illustrations	32
1.4.c Radius of convergence of a power series	34
1.4.d Analytic functions	35
1.5 A quick look at asymptotic and divergent series	37
1.5.a Asymptotic series	37
1.5.b Divergent series and asymptotic expansions	38
<i>Exercises</i>	43
<i>Problem</i>	46
<i>Solutions</i>	47
2 Measure theory and the Lebesgue integral	51
2.1 The integral according to Mr. Riemann	51
2.1.a Riemann sums	51
2.1.b Limitations of Riemann's definition	54
2.2 The integral according to Mr. Lebesgue	54
2.2.a Principle of the method	55

2.2.b	Borel subsets	56
2.2.c	Lebesgue measure	58
2.2.d	The Lebesgue σ -algebra	59
2.2.e	Negligible sets	61
2.2.f	Lebesgue measure on \mathbb{R}^n	62
2.2.g	Definition of the Lebesgue integral	62
2.2.h	Functions zero almost everywhere, space L^1	66
2.2.i	And today?	67
<i>Exercises</i>		68
<i>Solutions</i>		71
3	Integral calculus	73
3.1	Integrability in practice	73
3.1.a	Standard functions	73
3.1.b	Comparison theorems	74
3.2	Exchanging integrals and limits or series	75
3.3	Integrals with parameters	77
3.3.a	Continuity of functions defined by integrals	77
3.3.b	Differentiating under the integral sign	78
3.3.c	Case of parameters appearing in the integration range	78
3.4	Double and multiple integrals	79
3.5	Change of variables	81
<i>Exercises</i>		83
<i>Solutions</i>		85
4	Complex Analysis I	87
4.1	Holomorphic functions	87
4.1.a	Definitions	88
4.1.b	Examples	90
4.1.c	The operators $\partial/\partial z$ and $\partial/\partial \bar{z}$	91
4.2	Cauchy's theorem	93
4.2.a	Path integration	93
4.2.b	Integrals along a circle	95
4.2.c	Winding number	96
4.2.d	Various forms of Cauchy's theorem	96
4.2.e	Application	99
4.3	Properties of holomorphic functions	99
4.3.a	The Cauchy formula and applications	99
4.3.b	Maximum modulus principle	104
4.3.c	Other theorems	105
4.3.d	Classification of zero sets of holomorphic functions	106
4.4	Singularities of a function	108
4.4.a	Classification of singularities	108
4.4.b	Meromorphic functions	110
4.5	Laurent series	111
4.5.a	Introduction and definition	111
4.5.b	Examples of Laurent series	113
4.5.c	The Residue theorem	114
4.5.d	Practical computations of residues	116

4.6	Applications to the computation of horrifying integrals or ghastly sums	117
4.6.a	Jordan's lemmas	117
4.6.b	Integrals on \mathbb{R} of a rational function	118
4.6.c	Fourier integrals	120
4.6.d	Integral on the unit circle of a rational function	121
4.6.e	Computation of infinite sums	122
<i>Exercises</i>		125
<i>Problem</i>		128
<i>Solutions</i>		129
5	Complex Analysis II	135
5.1	Complex logarithm; multivalued functions	135
5.1.a	The complex logarithms	135
5.1.b	The square root function	137
5.1.c	Multivalued functions, Riemann surfaces	137
5.2	Harmonic functions	139
5.2.a	Definitions	139
5.2.b	Properties	140
5.2.c	A trick to find f knowing u	142
5.3	Analytic continuation	144
5.4	Singularities at infinity	146
5.5	The saddle point method	148
5.5.a	The general saddle point method	149
5.5.b	The real saddle point method	152
<i>Exercises</i>		153
<i>Solutions</i>		154
6	Conformal maps	155
6.1	Conformal maps	155
6.1.a	Preliminaries	155
6.1.b	The Riemann mapping theorem	157
6.1.c	Examples of conformal maps	158
6.1.d	The Schwarz-Christoffel transformation	161
6.2	Applications to potential theory	163
6.2.a	Application to electrostatics	165
6.2.b	Application to hydrodynamics	167
6.2.c	Potential theory, lightning rods, and percolation	169
6.3	Dirichlet problem and Poisson kernel	170
<i>Exercises</i>		174
<i>Solutions</i>		176
7	Distributions I	179
7.1	Physical approach	179
7.1.a	The problem of distribution of charge	179
7.1.b	The problem of momentum and forces during an elastic shock	181
7.2	Definitions and examples of distributions	182
7.2.a	Regular distributions	184
7.2.b	Singular distributions	185
7.2.c	Support of a distribution	187

7.2.d	Other examples	187
7.3	Elementary properties. Operations	188
7.3.a	Operations on distributions	188
7.3.b	Derivative of a distribution	191
7.4	Dirac and its derivatives	193
7.4.a	The Heaviside distribution	193
7.4.b	Multidimensional Dirac distributions	194
7.4.c	The distribution δ'	196
7.4.d	Composition of δ with a function	198
7.4.e	Charge and current densities	199
7.5	Derivation of a discontinuous function	201
7.5.a	Derivation of a function discontinuous at a point	201
7.5.b	Derivative of a function with discontinuity along a surface \mathcal{S}	204
7.5.c	Laplacian of a function discontinuous along a surface \mathcal{S}	206
7.5.d	Application: laplacian of $1/r$ in 3-space	207
7.6	Convolution	209
7.6.a	The tensor product of two functions	209
7.6.b	The tensor product of distributions	209
7.6.c	Convolution of two functions	211
7.6.d	“Fuzzy” measurement	213
7.6.e	Convolution of distributions	214
7.6.f	Applications	215
7.6.g	The Poisson equation	216
7.7	Physical interpretation of convolution operators	217
7.8	Discrete convolution	220
8	Distributions II	223
8.1	Cauchy principal value	223
8.1.a	Definition	223
8.1.b	Application to the computation of certain integrals	224
8.1.c	Feynman’s notation	225
8.1.d	Kramers-Kronig relations	227
8.1.e	A few equations in the sense of distributions	229
8.2	Topology in \mathcal{D}'	230
8.2.a	Weak convergence in \mathcal{D}'	230
8.2.b	Sequences of functions converging to δ	231
8.2.c	Convergence in \mathcal{D}' and convergence in the sense of functions	234
8.2.d	Regularization of a distribution	234
8.2.e	Continuity of convolution	235
8.3	Convolution algebras	236
8.4	Solving a differential equation with initial conditions	238
8.4.a	First order equations	238
8.4.b	The case of the harmonic oscillator	239
8.4.c	Other equations of physical origin	240
	<i>Exercises</i>	241
	<i>Problem</i>	244
	<i>Solutions</i>	245

9	Hilbert spaces; Fourier series	249
9.1	Insufficiency of vector spaces	249
9.2	Pre-Hilbert spaces	251
9.2.a	The finite-dimensional case	254
9.2.b	Projection on a finite-dimensional subspace	254
9.2.c	Bessel inequality	256
9.3	Hilbert spaces	256
9.3.a	Hilbert basis	257
9.3.b	The ℓ^2 space	261
9.3.c	The space $L^2[0, a]$	262
9.3.d	The $L^2(\mathbb{R})$ space	263
9.4	Fourier series expansion	264
9.4.a	Fourier coefficients of a function	264
9.4.b	Mean-square convergence	265
9.4.c	Fourier series of a function $f \in L^1[0, a]$	266
9.4.d	Pointwise convergence of the Fourier series	267
9.4.e	Uniform convergence of the Fourier series	269
9.4.f	The Gibbs phenomenon	270
	<i>Exercises</i>	270
	<i>Problem</i>	271
	<i>Solutions</i>	272
10	Fourier transform of functions	277
10.1	Fourier transform of a function in L^1	277
10.1.a	Definition	278
10.1.b	Examples	279
10.1.c	The L^1 space	279
10.1.d	Elementary properties	280
10.1.e	Inversion	282
10.1.f	Extension of the inversion formula	284
10.2	Properties of the Fourier transform	285
10.2.a	Transpose and translates	285
10.2.b	Dilation	286
10.2.c	Derivation	286
10.2.d	Rapidly decaying functions	288
10.3	Fourier transform of a function in L^2	288
10.3.a	The space \mathcal{S}	289
10.3.b	The Fourier transform in L^2	290
10.4	Fourier transform and convolution	292
10.4.a	Convolution formula	292
10.4.b	Cases of the convolution formula	293
	<i>Exercises</i>	295
	<i>Solutions</i>	296
11	Fourier transform of distributions	299
11.1	Definition and properties	299
11.1.a	Tempered distributions	300
11.1.b	Fourier transform of tempered distributions	301
11.1.c	Examples	303

11.1.d	Higher-dimensional Fourier transforms	305
11.1.e	Inversion formula	306
11.2	The Dirac comb	307
11.2.a	Definition and properties	307
11.2.b	Fourier transform of a periodic function	308
11.2.c	Poisson summation formula	309
11.2.d	Application to the computation of series	310
11.3	The Gibbs phenomenon	311
11.4	Application to physical optics	314
11.4.a	Link between diaphragm and diffraction figure	314
11.4.b	Diaphragm made of infinitely many infinitely narrow slits	315
11.4.c	Finite number of infinitely narrow slits	316
11.4.d	Finitely many slits with finite width	318
11.4.e	Circular lens	320
11.5	Limitations of Fourier analysis and wavelets	321
	<i>Exercises</i>	324
	<i>Problem</i>	325
	<i>Solutions</i>	326
12	The Laplace transform	331
12.1	Definition and integrability	331
12.1.a	Definition	332
12.1.b	Integrability	333
12.1.c	Properties of the Laplace transform	336
12.2	Inversion	336
12.3	Elementary properties and examples of Laplace transforms	338
12.3.a	Translation	338
12.3.b	Convolution	339
12.3.c	Differentiation and integration	339
12.3.d	Examples	341
12.4	Laplace transform of distributions	342
12.4.a	Definition	342
12.4.b	Properties	342
12.4.c	Examples	344
12.4.d	The z-transform	344
12.4.e	Relation between Laplace and Fourier transforms	345
12.5	Physical applications, the Cauchy problem	346
12.5.a	Importance of the Cauchy problem	346
12.5.b	A simple example	347
12.5.c	Dynamics of the electromagnetic field without sources	348
	<i>Exercises</i>	351
	<i>Solutions</i>	352
13	Physical applications of the Fourier transform	355
13.1	Justification of sinusoidal regime analysis	355
13.2	Fourier transform of vector fields: longitudinal and transverse fields	358
13.3	Heisenberg uncertainty relations	359
13.4	Analytic signals	365
13.5	Autocorrelation of a finite energy function	368

13.5.a	Definition	368
13.5.b	Properties	368
13.5.c	Intercorrelation	369
13.6	Finite power functions	370
13.6.a	Definitions	370
13.6.b	Autocorrelation	370
13.7	Application to optics: the Wiener-Khintchine theorem	371
	<i>Exercises</i>	375
	<i>Solutions</i>	376
14	Bras, kets, and all that sort of thing	377
14.1	Reminders about finite dimension	377
14.1.a	Scalar product and representation theorem	377
14.1.b	Adjoint	378
14.1.c	Symmetric and hermitian endomorphisms	379
14.2	Kets and bras	379
14.2.a	Kets $ \psi\rangle \in H$	379
14.2.b	Bras $\langle\psi \in H'$	380
14.2.c	Generalized bras	382
14.2.d	Generalized kets	383
14.2.e	$\text{Id} = \sum_n \varphi_n\rangle\langle\varphi_n $	384
14.2.f	Generalized basis	385
14.3	Linear operators	387
14.3.a	Operators	387
14.3.b	Adjoint	389
14.3.c	Bounded operators, closed operators, closable operators	390
14.3.d	Discrete and continuous spectra	391
14.4	Hermitian operators; self-adjoint operators	393
14.4.a	Definitions	394
14.4.b	Eigenvectors	396
14.4.c	Generalized eigenvectors	397
14.4.d	“Matrix” representation	398
14.4.e	Summary of properties of the operators P and X	401
	<i>Exercises</i>	403
	<i>Solutions</i>	404
15	Green functions	407
15.1	Generalities about Green functions	407
15.2	A pedagogical example: the harmonic oscillator	409
15.2.a	Using the Laplace transform	410
15.2.b	Using the Fourier transform	410
15.3	Electromagnetism and the d’Alembertian operator	414
15.3.a	Computation of the advanced and retarded Green functions	414
15.3.b	Retarded potentials	418
15.3.c	Covariant expression of advanced and retarded Green functions	421
15.3.d	Radiation	421
15.4	The heat equation	422
15.4.a	One-dimensional case	423
15.4.b	Three-dimensional case	426

15.5	Quantum mechanics	427
15.6	Klein-Gordon equation	429
	<i>Exercises</i>	432
16	Tensors	433
16.1	Tensors in affine space	433
16.1.a	Vectors	433
16.1.b	Einstein convention	435
16.1.c	Linear forms	436
16.1.d	Linear maps	438
16.1.e	Lorentz transformations	439
16.2	Tensor product of vector spaces: tensors	439
16.2.a	Existence of the tensor product of two vector spaces	439
16.2.b	Tensor product of linear forms: tensors of type $\binom{0}{2}$	441
16.2.c	Tensor product of vectors: tensors of type $\binom{2}{0}$	443
16.2.d	Tensor product of a vector and a linear form: linear maps or $\binom{1}{1}$ -tensors	444
16.2.e	Tensors of type $\binom{p}{q}$	446
16.3	The metric, <i>or</i> , how to raise and lower indices	447
16.3.a	Metric and pseudo-metric	447
16.3.b	Natural duality by means of the metric	449
16.3.c	Gymnastics: raising and lowering indices	450
16.4	Operations on tensors	453
16.5	Change of coordinates	455
16.5.a	Curvilinear coordinates	455
16.5.b	Basis vectors	456
16.5.c	Transformation of physical quantities	458
16.5.d	Transformation of linear forms	459
16.5.e	Transformation of an arbitrary tensor field	460
16.5.f	Conclusion	461
	<i>Solutions</i>	462
17	Differential forms	463
17.1	Exterior algebra	463
17.1.a	1-forms	463
17.1.b	Exterior 2-forms	464
17.1.c	Exterior k -forms	465
17.1.d	Exterior product	467
17.2	Differential forms on a vector space	469
17.2.a	Definition	469
17.2.b	Exterior derivative	470
17.3	Integration of differential forms	471
17.4	Poincaré's theorem	474
17.5	Relations with vector calculus: gradient, divergence, curl	476
17.5.a	Differential forms in dimension 3	476
17.5.b	Existence of the scalar electrostatic potential	477
17.5.c	Existence of the vector potential	479
17.5.d	Magnetic monopoles	480

17.6	Electromagnetism in the language of differential forms	480
	<i>Problem</i>	484
	<i>Solution</i>	485
18	Groups and group representations	489
18.1	Groups	489
18.2	Linear representations of groups	491
18.3	Vectors and the group $SO(3)$	492
18.4	The group $SU(2)$ and spinors	497
18.5	Spin and Riemann sphere	503
	<i>Exercises</i>	505
19	Introduction to probability theory	509
19.1	Introduction	510
19.2	Basic definitions	512
19.3	Poincaré formula	516
19.4	Conditional probability	517
19.5	Independent events	519
20	Random variables	521
20.1	Random variables and probability distributions	521
20.2	Distribution function and probability density	524
20.2.a	Discrete random variables	526
20.2.b	(Absolutely) continuous random variables	526
20.3	Expectation and variance	527
20.3.a	Case of a discrete r.v.	527
20.3.b	Case of a continuous r.v.	528
20.4	An example: the Poisson distribution	530
20.4.a	Particles in a confined gas	530
20.4.b	Radioactive decay	531
20.5	Moments of a random variable	532
20.6	Random vectors	534
20.6.a	Pair of random variables	534
20.6.b	Independent random variables	537
20.6.c	Random vectors	538
20.7	Image measures	539
20.7.a	Case of a single random variable	539
20.7.b	Case of a random vector	540
20.8	Expectation and characteristic function	540
20.8.a	Expectation of a function of random variables	540
20.8.b	Moments, variance	541
20.8.c	Characteristic function	541
20.8.d	Generating function	543
20.9	Sum and product of random variables	543
20.9.a	Sum of random variables	543
20.9.b	Product of random variables	546
20.9.c	Example: Poisson distribution	547
20.10	Bienaymé-Tchebychev inequality	547
20.10.a	Statement	547

20.10.b Application: Buffon's needle 549
 20.11 Independence, correlation, causality 550

21 Convergence of random variables: central limit theorem 553
 21.1 Various types of convergence 553
 21.2 The law of large numbers 555
 21.3 Central limit theorem 556
Exercises 560
Problems 563
Solutions 564

Appendices

A Reminders concerning topology and normed vector spaces 573
 A.1 Topology, topological spaces 573
 A.2 Normed vector spaces 577
 A.2.a Norms, seminorms 577
 A.2.b Balls and topology associated to the distance 578
 A.2.c Comparison of sequences 580
 A.2.d Bolzano-Weierstrass theorems 581
 A.2.e Comparison of norms 581
 A.2.f Norm of a linear map 583
Exercise 583
Solution 584

B Elementary reminders of differential calculus 585
 B.1 Differential of a real-valued function 585
 B.1.a Functions of one real variable 585
 B.1.b Differential of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ 586
 B.1.c Tensor notation 587
 B.2 Differential of map with values in \mathbb{R}^p 587
 B.3 Lagrange multipliers 588
Solution 591

C Matrices 593
 C.1 Duality 593
 C.2 Application to matrix representation 594
 C.2.a Matrix representing a family of vectors 594
 C.2.b Matrix of a linear map 594
 C.2.c Change of basis 595
 C.2.d Change of basis formula 595
 C.2.e Case of an orthonormal basis 596

D A few proofs 597

Tables

Fourier transforms 609
Laplace transforms 613
Probability laws 616

Further reading 617
References 621
Portraits 627
Sidebars 629
Index 631