Contents

Preface *page* xvii

Part I Plasma Physics Preliminaries 1

1 Introduction 3
1.1 Motivation 3
1.2 Thermonuclear fusion and plasma confinement 4
1.2.1 Fusion reactions 4
1.2.2 Conditions for fusion 6
1.2.3 Magnetic confinement and tokamaks 9
1.3 Astrophysical plasmas 11
1.3.1 Celestial mechanics 11
1.3.2 Astrophysics 13
1.3.3 Plasmas enter the stage 15
1.3.4 The standard view of nature 17
1.4 Definitions of the plasma state 19
1.4.1 Microscopic definition of plasma 19
1.4.2 Macroscopic approach to plasma 23
1.5 Literature and exercises 24

2 Elements of plasma physics 27
2.1 Theoretical models 27
2.2 Single particle motion 27
2.2.1 Cyclotron motion 27
2.2.2 Excursion: Basic equations of electrodynamics and mechanics 30
2.2.3 Drifts, adiabatic invariants 33
2.3 Kinetic plasma theory 38
2.3.1 Boltzmann equation and moment reduction 38
2.3.2 Collective phenomena: plasma oscillations 43
2.3.3 Landau damping 46
2.4 Fluid description 52
2.4.1 From the two-fluid to the MHD description of plasmas 53
2.4.2 Alfvén waves 57
2.4.3 Equilibrium and stability 59

2.5 In conclusion 63
2.6 Literature and exercises 64

3 'Derivation' of the macroscopic equations★ 66
3.1 Two approaches★ 66
3.2 Kinetic equations★ 67
3.2.1 Boltzmann equation★ 67
3.2.2 Moments of the Boltzmann equation★ 70
3.2.3 Thermal fluctuations and transport★ 72
3.2.4 Collisions and closure★ 75
3.3 Two-fluid equations★ 78
3.3.1 Electron–ion plasma★ 78
3.3.2 The classical transport coefficients★ 79
3.3.3 Dissipative versus ideal fluids★ 83
3.3.4 Excursion: waves in two-fluid plasmas★ 86
3.4 One-fluid equations★ 95
3.4.1 Maximal ordering for MHD★ 95
3.4.2 Resistive and ideal MHD equations★ 99
3.5 Literature and exercises★ 101

Part II Basic Magnetohydrodynamics 103

4 The MHD model 105
4.1 The ideal MHD equations 105
4.1.1 Postulating the basic equations 105
4.1.2 Scale independence 110
4.1.3 A crucial question 112
4.2 Magnetic flux 113
4.2.1 Flux tubes 113
4.2.2 Global magnetic flux conservation 114
4.3 Conservation laws 116
4.3.1 Conservation form of the MHD equations 116
4.3.2 Global conservation laws 118
4.3.3 Local conservation of magnetic flux 121
4.3.4 Magnetic helicity 124
4.4 Dissipative magnetohydrodynamics 128
4.4.1 Resistive MHD 128
4.4.2 (Non-)conservation form of the dissipative equations★ 131
4.5 Discontinuities 133
4.5.1 Shocks and jump conditions 133
4.5.2 Boundary conditions for plasmas with an interface 136
4.6 Model problems 138
4.6.1 Laboratory plasmas (models I–III) 138
4.6.2 Energy conservation for interface plasmas 141
4.6.3 Astrophysical plasmas (models IV–VI) 143

4.7 Literature and exercises 144

5 Waves and characteristics 147
5.1 Physics and accounting 147
5.1.1 Introduction 147
5.1.2 Sound waves 147
5.2 MHD waves 150
5.2.1 Symmetric representation in primitive variables 150
5.2.2 Entropy wave and magnetic field constraint 152
5.2.3 Reduction to velocity representation: three waves 155
5.2.4 Dispersion diagrams 157
5.3 Phase and group diagrams 159
5.3.1 Basic concepts 159
5.3.2 Application to the MHD waves 161
5.3.3 Asymptotic properties 165
5.3.4 Self-gravity and contraction in homogeneous media★ 166
5.4 Characteristics★ 169
5.4.1 The method of characteristics★ 169
5.4.2 Classification of partial differential equations★ 171
5.4.3 Characteristics in ideal MHD★ 173
5.5 Literature and exercises 179

6 Spectral theory 181
6.1 Stability: intuitive approach 181
6.1.1 Two viewpoints 181
6.1.2 Linearization and Lagrangian reduction 183
6.2 Force operator formalism 186
6.2.1 Equation of motion 186
6.2.2 Hilbert space 190
6.2.3 Proof of self-adjointness of the force operator 191
6.3 Spectral alternatives★ 196
6.3.1 Mathematical intermezzo★ 196
6.3.2 Initial value problem in MHD★ 198
6.4 Quadratic forms and variational principles 200
6.4.1 Expressions for the potential energy 200
6.4.2 Hamilton's principle 202
6.4.3 Rayleigh–Ritz spectral variational principle 203
6.4.4 Energy principle 204
6.5 Further spectral issues 206
6.5.1 Normal modes and the energy principle★ 206
6.5.2 Proof of the energy principle★ 207
6.5.3 σ-stability 209
6.5.4 Returning to the two viewpoints 210
6.6 Extension to interface plasmas 213
6.6.1 Boundary conditions at the interface 215

6.6.2 Self-adjointness for interface plasmas 218
6.6.3 Extended variational principles 219
6.6.4 Application to the Rayleigh–Taylor instability 221
6.7 Literature and exercises 229

Part III Standard Model Applications 231

7 Waves and instabilities of inhomogeneous plasmas 233
7.1 Hydrodynamics of the solar interior 233
7.1.1 Radiative equilibrium model 234
7.1.2 Convection zone 237
7.2 Hydrodynamic waves and instabilities of a gravitating slab 239
7.2.1 Hydrodynamic wave equation 239
7.2.2 Convective instabilities 241
7.2.3 Gravito-acoustic waves 242
7.2.4 Helioseismology and MHD spectroscopy 245
7.3 MHD wave equation for a gravitating magnetized plasma slab 248
7.3.1 Preliminaries 248
7.3.2 MHD wave equation for a gravitating slab 252
7.3.3 Gravito-MHD waves 258
7.4 Continuous spectrum and spectral structure 265
7.4.1 Singular differential equations 265
7.4.2 Alfvén and slow continua 269
7.4.3 Oscillation theorems 273
7.4.4 Cluster spectra* 278
7.5 Gravitational instabilities of a magnetized plasma slab 279
7.5.1 Energy principle for a gravitating plasma slab 280
7.5.2 Interchanges in shearless magnetic fields 283
7.5.3 Interchange instabilities in sheared magnetic fields 285
7.6 Literature and exercises 289

8 Magnetic structures and dynamics of the solar system 292
8.1 Plasma dynamics in laboratory and nature 292
8.2 Solar magnetism 293
8.2.1 The solar cycle 294
8.2.2 Magnetic structures in the solar atmosphere 300
8.2.3 Inspiration from solar magnetism 309
8.2.4 Solar wind and heliosphere 309
8.3 Space weather 313
8.3.1 Technological and economic implications 313
8.3.2 Coronal mass ejections 314
8.3.3 Numerical modelling of space weather 317
8.3.4 Solar wind and planetary magnetospheres 320
8.4 Perspective 321
8.5 Literature and exercises 322

9 Cylindrical plasmas 325
9.1 Equilibrium of cylindrical plasmas 325
9.1.1 Diffuse plasmas 325
9.1.2 Interface plasmas 329
9.2 MHD wave equation for cylindrical plasmas 330
9.2.1 Derivation of the MHD wave equation for a cylinder 330
9.2.2 Boundary conditions for cylindrical interfaces 336
9.3 Spectral structure 339
9.3.1 One-dimensional inhomogeneity 339
9.3.2 Cylindrical model problems 341
9.3.3 Cluster spectra* 347
9.4 Stability of cylindrical plasmas 348
9.4.1 Oscillation theorems for stability 348
9.4.2 Stability of plasmas with shearless magnetic fields 353
9.4.3 Stability of force-free magnetic fields* 357
9.4.4 Stability of the 'straight tokamak' 361
9.5 Literature and exercises 369

10 Initial value problem and wave damping* 372
10.1 Implications of the continuous spectrum* 372
10.2 Initial value problem* 373
10.2.1 Reduction to a one-dimensional representation* 373
10.2.2 Restoring the three-dimensional picture* 376
10.3 Damping of Alfvén waves* 380
10.3.1 Green's function* 381
10.3.2 Spectral cuts* 384
10.4 Quasi-modes* 386
10.5 Leaky modes* 392
10.6 Literature and exercises* 397

11 Resonant absorption and wave heating 399
11.1 Ideal MHD theory of resonant absorption 399
11.1.1 Analytical solution of a simple model problem 399
11.1.2 Role of the singularity 405
11.1.3 Resonant 'absorption' versus resonant 'dissipation' 414
11.2 Heating and wave damping in tokamaks and coronal loops 417
11.2.1 Tokamaks 417
11.2.2 Coronal loops and arcades 418
11.2.3 Numerical analysis of resonant absorption 419
11.3 Alternative excitation mechanisms 423
11.3.1 Foot point driving 424
11.3.2 Phase mixing 427
11.3.3 Applications to solar and magnetospheric plasmas 428
11.4 Literature and exercises 432

Part IV Flow and Dissipation 435

12 Waves and instabilities of stationary plasmas 437
12.1 Laboratory and astrophysical plasmas 437
12.1.1 Grand vision: magnetized plasma on all scales! 437
12.1.2 Laboratory and astrophysical plasmas 440
12.1.3 Interchanges and the Parker instability 441
12.2 Spectral theory of stationary plasmas 445
12.2.1 Plasmas with background flow 445
12.2.2 Frieman–Rotenberg formulation 448
12.2.3 Self-adjointness of the generalized force operator* 453
12.2.4 Energy conservation and stability 456
12.3 The Spectral Web 462
12.3.1 Opening up the boundaries 462
12.3.2 Oscillation theorems in the complex plane 466
12.4 Literature and exercises 471

13 Shear flow and rotation 473
13.1 Spectral theory of plane plasmas with shear flow 473
13.1.1 Gravito-MHD wave equation for plane plasma flow 473
13.1.2 Kelvin–Helmholtz instabilities in interface plasmas 478
13.1.3 Continua and the real oscillation theorem 480
13.1.4 Spectral Web and the complex oscillation theorem 484
13.2 Analysis of flow-driven instabilities in plane plasmas 486
13.2.1 Rayleigh–Taylor instabilities of magnetized plasmas 488
13.2.2 Kelvin–Helmholtz instabilities of ordinary fluids 489
13.2.3 Combined instabilities of magnetized plasmas 494
13.3 Spectral theory of rotating plasmas 498
13.3.1 MHD wave equation for cylindrical flow in 3D 498
13.3.2 Reduction to a second order differential equation 500
13.3.3 Singular expansions* 502
13.3.4 Doppler–Coriolis shift and solution path 505
13.4 Rayleigh–Taylor instabilities in rotating theta-pinches 506
13.4.1 Hydrodynamic modes ($k = 0$) 507
13.4.2 Magnetohydrodynamic modifications ($k \neq 0$) 511
13.5 Magneto-rotational instability in accretion discs 513
13.5.1 Analytical preliminaries 514
13.5.2 Numerical Spectral Web solutions 518
13.6 Literature and exercises 523

14 Resistive plasma dynamics 525
14.1 Plasmas with dissipation 525
14.1.1 Conservative versus dissipative dynamical systems 525
14.1.2 Stability of force-free magnetic fields: a trap 525
14.2 Resistive instabilities 532
14.2.1 Basic equations 532

14.2.2 Tearing modes 534
14.2.3 Resistive interchange modes 543
14.3 Resistive spectrum 544
14.3.1 Resistive wall mode 544
14.3.2 Spectrum of homogeneous plasma 548
14.3.3 Spectrum of inhomogeneous plasma 551
14.4 Reconnection 554
14.4.1 Reconnection in a 2D Harris sheet 554
14.4.2 Petschek reconnection 558
14.4.3 Kelvin–Helmholtz induced tearing instabilities 559
14.4.4 Extended MHD and reconnection 560
14.5 Excursion: Hall-MHD wave diagrams 563
14.6 Literature and exercises 566

15 Computational linear MHD 569
15.1 Spatial discretization techniques 569
15.1.1 Basic concepts for discrete representations 571
15.1.2 Finite difference methods 572
15.1.3 Finite element method 576
15.1.4 Spectral methods 583
15.1.5 Mixed representations 586
15.2 Linear MHD: boundary value problems 588
15.2.1 Linearized MHD equations 589
15.2.2 Steady solutions to linearly driven problems 590
15.2.3 MHD eigenvalue problems 593
15.2.4 Extended MHD examples 594
15.3 Linear MHD: initial value problems 599
15.3.1 Temporal discretizations: explicit methods 599
15.3.2 Disparateness of MHD time scales 606
15.3.3 Temporal discretizations: implicit methods 606
15.3.4 Applications: linear MHD evolutions 608
15.4 Concluding remarks 612
15.5 Literature and exercises 612

Part V Toroidal Geometry 615

16 Static equilibrium of toroidal plasmas 617
16.1 Axi-symmetric equilibrium 617
16.1.1 Equilibrium in tokamaks 617
16.1.2 Magnetic field geometry 621
16.1.3 Cylindrical limits 624
16.1.4 Global confinement and parameters 627
16.2 Grad–Shafranov equation 635
16.2.1 Derivation of the Grad–Shafranov equation 635
16.2.2 Large aspect ratio expansion: internal solution 637

16.2.3 Large aspect ratio expansion: external solution 642
16.3 Exact equilibrium solutions 647
16.3.1 Poloidal flux scaling 647
16.3.2 Soloviev equilibrium 652
16.3.3 Numerical equilibria* 655
16.4 Extensions 660
16.4.1 Toroidal rotation 660
16.4.2 Gravitating plasma equilibria* 662
16.4.3 Challenges 663
16.5 Literature and exercises 664

17 Linear dynamics of static toroidal plasmas 667
17.1 "Ad more geometrico" 667
17.1.1 Alfvén wave dynamics in toroidal geometry 667
17.1.2 Coordinates and mapping 667
17.1.3 Geometrical–physical characteristics 668
17.2 Analysis of waves and instabilities in toroidal geometry 674
17.2.1 Spectral wave equation 674
17.2.2 Spectral variational principle 676
17.2.3 Alfvén and slow continuum modes 677
17.2.4 Poloidal mode coupling 680
17.2.5 Alfvén and slow ballooning modes 683
17.3 Computation of waves and instabilities in tokamaks 690
17.3.1 Ideal MHD versus resistive MHD in computations 690
17.3.2 Internal modes 695
17.3.3 Edge localized modes 697
17.3.4 Toroidal Alfvén eigenmodes and MHD spectroscopy 701
17.4 Literature and exercises 704

18 Linear dynamics of toroidal plasmas with flow* 707
18.1 Transonic toroidal plasmas 707
18.2 Axi-symmetric equilibrium of transonic stationary states* 709
18.2.1 Equilibrium flux functions* 709
18.2.2 Equilibrium variational principle and rescaling* 712
18.2.3 Elliptic and hyperbolic flow regimes* 715
18.2.4 Expansion of the equilibrium in small toroidicity* 716
18.3 Equations for the continuous spectrum* 722
18.3.1 Reduction for straight-field-line coordinates* 722
18.3.2 Continua of poloidally and toroidally rotating plasmas* 725
18.3.3 Analysis of trans-slow continua for small toroidicity* 731
18.4 Trans-slow continua in tokamaks and accretion discs* 737
18.4.1 Tokamaks and magnetically dominated accretion discs* 738
18.4.2 Gravity dominated accretion discs* 740
18.4.3 Trans-slow Alfvén continuum instabilities 742
18.5 Literature and exercises* 744

Part VI Nonlinear Dynamics 747

19 Turbulence in incompressible magneto-fluids 749
19.1 Incompressible hydrodynamics preliminaries 749
19.1.1 The incompressible hydro model 749
19.1.2 Two-dimensional formulations 751
19.1.3 'Wave' analysis for incompressible Euler 751
19.1.4 Energy equation and Kolmogorov scaling 753
19.1.5 Selected numerical examples 756
19.2 Incompressible magnetohydrodynamics 758
19.2.1 Governing equations 758
19.2.2 Elsässer formulation 759
19.2.3 Kinematic MHD modelling 760
19.2.4 Dynamo aspects 761
19.3 Waves in incompressible MHD 764
19.3.1 Linear wave analysis 765
19.3.2 Nonlinear wave solutions and conservation laws 766
19.3.3 MHD turbulence scaling laws 768
19.4 Incompressible MHD simulations 771
19.4.1 Structure formation in incompressible MHD studies 772
19.4.2 Dynamo aspects continued 774
19.5 Extension to compressible MHD and concluding remarks 776
19.6 Literature and exercises 778

20 Computational nonlinear MHD 780
20.1 General considerations for nonlinear conservation laws 780
20.1.1 Conservative versus primitive variable formulations 780
20.1.2 Scalar conservation law and the Riemann problem 786
20.1.3 Numerical discretizations for scalar conservation 790
20.1.4 Finite volume treatments 796
20.2 Upwind-like finite volume treatments for one-dimensional MHD 797
20.2.1 The Godunov method 798
20.2.2 A robust shock-capturing method: TVDLF 802
20.2.3 Approximate Riemann solver schemes 807
20.2.4 Simulating 1D MHD Riemann problems 811
20.3 Multi-dimensional MHD computations 813
20.3.1 $\nabla \cdot \mathbf{B} = 0$ condition for shock-capturing schemes 814
20.3.2 Example nonlinear MHD scenarios 819
20.3.3 Alternative numerical methods 822
20.4 Implicit approaches for extended MHD simulations 827
20.4.1 Semi-implicit methods 828
20.4.2 Simulating ideal and resistive instabilities 832
20.4.3 Global simulations for tokamak plasmas 833
20.5 Literature and exercises 834

21 Transonic MHD flows and shocks 837
21.1 Transonic flows 837
21.1.1 Characteristics and shocks 838
21.1.2 Gas dynamic shocks 840
21.1.3 Misnomers 845
21.2 MHD shock conditions 846
21.2.1 MHD discontinuities without mass flow 846
21.2.2 MHD discontinuities with mass flow 848
21.2.3 Slow, intermediate and fast shocks 852
21.3 Advanced classification of MHD shocks 854
21.3.1 Distilled shock conditions 854
21.3.2 Time reversal duality 859
21.3.3 Angular dependence of MHD shocks* 865
21.3.4 Observational considerations of MHD shocks 870
21.4 Example astrophysical transonic flows 871
21.5 Literature and exercises 876

22 Ideal MHD in special relativity 879
22.1 Four-dimensional space-time: special relativistic concepts 879
22.1.1 Space-time coordinates and Lorentz transformations 880
22.1.2 Four-vectors in flat space-time and invariants 882
22.1.3 Relativistic gas dynamics and stress-energy tensor 885
22.1.4 Sound waves and shock relations in relativistic gases 889
22.2 Electromagnetism and special relativistic MHD 895
22.2.1 Electromagnetic field tensor and Maxwell's equations 895
22.2.2 Ideal MHD in special relativity 900
22.2.3 Wave dynamics in a homogeneous plasma 902
22.2.4 Shock conditions in relativistic MHD 906
22.3 Computing relativistic magnetized plasma dynamics 908
22.3.1 Numerical challenges from relativistic MHD 910
22.3.2 Pulsar Wind Nebulae modelling 911
22.4 Outlook: General relativistic MHD simulations 915
22.5 Literature and exercises 916

Appendices 919

A Vectors and coordinates 919
A.1 Vector identities 919
A.2 Vector expressions in orthogonal coordinates 920
A.3 Vector expressions in non-orthogonal coordinates

B Tables of physical quantities

References 937
Index 964