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$ax + b$ and $cx + d$	31
$a + x$ and $b + x$	35
$ax^2 + bx + c$	40
$a^2 \pm x^2$	54
$a^3 \pm x^3$	80
$a^4 \pm x^4$	94
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