

# Contents

|   |    |
|---|----|
| <b>1 Delay Differential Equations .....</b>   | 1  |
| 1.1 Introduction .....  | 1  |
| 1.1.1 DDE with Single Constant Delay .....  | 3  |
| 1.1.2 DDE with Discrete Delays .....  | 4  |
| 1.1.3 DDE with Distributed Delay .....  | 5  |
| 1.1.4 DDE with State-Dependent Delay .....  | 6  |
| 1.1.5 DDE with Time-Dependent Delay .....   | 6  |
| 1.2 Constructing the Solution for DDEs with Single Constant Delay ..                | 7  |
| 1.2.1 Linear Delay Differential Equation .....                                      | 8  |
| 1.2.2 Numerical Simulation of DDEs .....  | 10 |
| 1.2.3 Nonlinear Delay Differential Equations .....                                  | 11 |
| 1.3 Salient Features of Chaotic Time-Delay Systems .....                            | 13 |
| References .....  | 13 |
| <br>  |    |
| <b>2 Linear Stability and Bifurcation Analysis .....</b>                            | 17 |
| 2.1 Introduction .....  | 17 |
| 2.2 Linear Stability Analysis .....   | 17 |
| 2.2.1 Example: Linear Delay Differential Equation .....                             | 19 |
| 2.3 A Geometric Approach to Study Stability .....                                   | 20 |
| 2.3.1 Example: Linear Delay Differential Equation .....                             | 21 |
| 2.4 A General Approach to Determine Linear Stability of Equilibrium<br>Points ..... | 22 |
| 2.4.1 Characteristic Equation .....   | 22 |
| 2.4.2 Stability Conditions .....  | 22 |
| 2.4.3 Stability Curves/Surfaces in the $(\tau, a, b)$ Parameter Space ..            | 23 |
| 2.4.4 Extension to Coupled DDEs/Complex Scalar DDEs .....                           | 24 |
| 2.4.5 Bifurcation Analysis .....  | 25 |
| 2.4.6 Results of Stability Analysis .....   | 25 |
| 2.4.7 A Theorem on the Stability of Equilibrium Points .....                        | 26 |
| 2.4.8 Example: Linear Delay Differential Equation .....                             | 26 |
| References .....  | 29 |

|  |    |
|--|----|
| <b>3 Bifurcation and Chaos in Time-Delayed Piecewise Linear Dynamical System</b> | 31 |
| 3.1 Introduction   | 31 |
| 3.2 Simple Scalar First Order Piecewise Linear DDE                               | 32 |
| 3.2.1 Fixed Points and Linear Stability  | 33 |
| 3.3 Numerical Study of the Single Scalar Piecewise Linear Time-Delay System      | 36 |
| 3.3.1 Dynamics in the Pseudospace  | 36 |
| 3.3.2 Transients   | 37 |
| 3.3.3 One and Two Parameter Bifurcation Diagrams                                 | 41 |
| 3.3.4 Lyapunov Exponents and Hyperchaotic Regimes                                | 43 |
| 3.4 Experimental Realization using PSPICE Simulation                             | 44 |
| 3.5 Stability Analysis and Chaotic Dynamics of Coupled DDEs                      | 46 |
| 3.5.1 Fixed Points and Linear Stability  | 46 |
| 3.6 Numerical Analysis of the Coupled DDE  | 49 |
| 3.6.1 Transients   | 50 |
| 3.6.2 One and Two Parameter Bifurcation Diagrams                                 | 51 |
| References   | 53 |
| <b>4 A Few Other Interesting Chaotic Delay Differential Equations</b>            | 55 |
| 4.1 Introduction   | 55 |
| 4.2 The Mackey-Glass System: A Typical Nonlinear DDE                             | 55 |
| 4.2.1 Mackey-Glass Time-Delay System   | 55 |
| 4.2.2 Fixed Points and Linear Stability Analysis                                 | 56 |
| 4.2.3 Time-Delay $\tau = 0$  | 57 |
| 4.2.4 Time-Delay $\tau > 0$  | 57 |
| 4.2.5 Numerical Simulation: Bifurcations and Chaos                               | 62 |
| 4.2.6 Experimental Realization Using Electronic Circuit                          | 64 |
| 4.3 Other Interesting Scalar Chaotic Time-Delay Systems                          | 67 |
| 4.3.1 A Simple Chaotic Delay Differential Equation                               | 67 |
| 4.3.2 Ikeda Time-Delay System  | 67 |
| 4.3.3 Scalar Time-Delay System with Polynomial Nonlinearity                      | 69 |
| 4.3.4 Scalar Time-Delay System with Other Piecewise Linear Nonlinearities        | 70 |
| 4.3.5 Another Form of Scalar Time-Delay System                                   | 73 |
| 4.3.6 El Niño and the Delayed Action Oscillator                                  | 76 |
| 4.4 Coupled Chaotic Time-Delay Systems   | 78 |
| 4.4.1 Time-Delayed Chua's Circuit  | 78 |
| 4.4.2 Semiconductor Lasers   | 79 |
| 4.4.3 Neural Networks  | 81 |
| References   | 82 |

|  |     |
|--|-----|
| <b>5 Implications of Delay Feedback: Amplitude Death and Other Effects</b>   | 85  |
| 5.1 Introduction   | 85  |
| 5.2 Time-Delay Induced Amplitude Death   | 85  |
| 5.2.1 Theoretical Study: Single Oscillator   | 86  |
| 5.2.2 Experimental Study   | 89  |
| 5.3 Amplitude Death with Distributed Delay in Coupled Limit Cycle Oscillators  | 91  |
| 5.4 Amplitude Death in Coupled Chaotic Oscillators   | 93  |
| 5.5 Amplitude Death with Conjugate (Dissimilar) Coupling   | 96  |
| 5.6 Amplitude Death with Dynamic Coupling  | 98  |
| 5.7 Time-Delay Induced Bifurcations  | 101 |
| 5.8 Some Other Effects of Delay Feedback   | 102 |
| References   | 103 |
| <b>6 Recent Developments on Delay Feedback/Coupling: Complex Networks, Chimeras, Globally Clustered Chimeras and Synchronization</b> | 105 |
| 6.1 Introduction   | 105 |
| 6.2 Complex Networks   | 105 |
| 6.3 Chimera States in Delay Coupled Identical Oscillators  | 108 |
| 6.3.1 Discovery of Chimera States  | 108 |
| 6.3.2 Chimera States in Delay Coupled Systems  | 111 |
| 6.4 Chimera States in Delay Coupled Subpopulations: Globally Clustered States  | 113 |
| 6.5 Synchronization in Complex Networks with Delay   | 117 |
| 6.6 Controlling Using Time-Delay Feedback  | 118 |
| 6.6.1 Pyragas Time-Delay Feedback Control  | 119 |
| 6.6.2 Transient Behavior with Time-Delay Feedback  | 122 |
| 6.7 Further Developments   | 124 |
| References   | 125 |
| <b>7 Complete Synchronization of Chaotic Oscillations in Coupled Time-Delay Systems</b>  | 127 |
| 7.1 Introduction   | 127 |
| 7.2 Complete Synchronization in Coupled Time-Delay Systems   | 129 |
| 7.3 Stability Using Krasovskii-Lyapunov Theory   | 130 |
| 7.4 Numerical Confirmation   | 133 |
| 7.4.1 Case 1   | 134 |
| 7.4.2 Case 2   | 134 |
| 7.4.3 Case 3   | 135 |
| 7.4.4 Case 4   | 135 |
| 7.5 Conclusion   | 135 |
| References   | 136 |

|  |     |
|--|-----|
| <b>8 Transition from Anticipatory to Lag Synchronization via Complete Synchronization</b>  | 139 |
| 8.1 Introduction   | 139 |
| 8.2 Coupled System and the General Stability Condition   | 139 |
| 8.3 Coupled Piecewise Linear Time-Delay System and Stability Condition: Transition from Anticipatory to Lag Synchronization                  | 141 |
| 8.3.1 Anticipatory Synchronization for $\tau_2 < \tau_1$   | 142 |
| 8.3.2 Complete Synchronization for $\tau_2 = \tau_1$   | 146 |
| 8.3.3 Lag Synchronization for $\tau_2 > \tau_1$  | 147 |
| 8.3.4 Inverse Synchronizations   | 149 |
| 8.4 Transition from Anticipatory to Lag via Complete Synchronization: Mackey-Glass and Ikeda Systems   | 153 |
| 8.4.1 Anticipatory Synchronization for $\tau_2 < \tau_1$   | 154 |
| 8.4.2 Complete Synchronization for $\tau_2 = \tau_1$   | 158 |
| 8.4.3 Lag Synchronization for $\tau_2 > \tau_1$  | 158 |
| 8.5 Inverse Synchronizations: Mackey-Glass and Ikeda Systems   | 161 |
| References   | 163 |
| <b>9 Intermittency Transition to Generalized Synchronization</b>   | 165 |
| 9.1 Introduction   | 165 |
| 9.2 Broad Range (Slow/Delayed) Intermittency Transition to GS for Linear Error Feedback Coupling of the Form $(x_1(t) - x_2(t))$             | 166 |
| 9.3 Stability Condition  | 167 |
| 9.4 Approximate (Intermittent) Generalized Synchronization   | 168 |
| 9.5 Characterization of IGS  | 171 |
| 9.6 Narrow Range (Immediate) Intermittency Transition to GS for Linear Direct Feedback Coupling of the Form $x_1(t)$                         | 173 |
| 9.7 Broad Range Intermittency Transition to GS for Nonlinear Error Feedback Coupling of the Form $(f(x_1(t - \tau_2)) - f(x_2(t - \tau_2)))$ | 178 |
| 9.8 Narrow Range Intermittency Transition to GS for Nonlinear Direct Feedback Coupling of the Form $f(x_1(t - \tau_2))$                      | 181 |
| 9.9 Intermittency Transition to Generalized Synchronization: Mackey-Glass & Ikeda Systems  | 185 |
| 9.9.1 Broad Range Intermittency Transition to GS   | 186 |
| 9.9.2 Narrow Range Intermittency Transition to GS  | 190 |
| References   | 199 |
| <b>10 Transition from Phase to Generalized Synchronization</b>   | 201 |
| 10.1 Introduction  | 201 |
| 10.2 Phase-Coherent and Non-phase-coherent Attractors  | 202 |
| 10.3 CPS in Chaotic Systems  | 203 |
| 10.4 CPS and Time-Delay Systems  | 205 |
| 10.5 CPS from Poincaré Surface of Section of the Transformed Attractor   | 207 |

|   |     |
|---|-----|
| 10.6 CPS from Recurrence Quantification Analysis  | 210 |
| 10.7 CPS from the Lyapunov Exponents  | 213 |
| 10.8 Concept of Localized Sets  | 214 |
| 10.9 Transition from Phase to Generalized Synchronization: Mackey-Glass & Ikeda Systems | 215 |
| 10.9.1 CPS from Poincaré Section of the Transformed Attractor                           | 217 |
| 10.9.2 CPS from Recurrence Quantification Analysis                                      | 218 |
| 10.9.3 CPS from the Lyapunov Exponents  | 219 |
| 10.9.4 CPS in Coupled Ikeda Systems   | 221 |
| 10.10 Summary   | 225 |
| References  | 225 |
| <b>11 DTM Induced Oscillating Synchronization</b>                                       | 227 |
| 11.1 Introduction   | 227 |
| 11.2 Estimation of the Effect of Delay Time Modulation                                  | 228 |
| 11.2.1 Filling Factor   | 228 |
| 11.2.2 Length of Polygon Line   | 230 |
| 11.2.3 Average Mutual Information   | 230 |
| 11.3 Coupled System and Stability Condition in the Presence of Delay Time Modulation    | 233 |
| 11.4 Oscillating Synchronization  | 235 |
| 11.5 Intermittent Anticipatory Synchronization  | 238 |
| 11.6 Complete Synchronization   | 241 |
| 11.7 Intermittent Lag Synchronization   | 242 |
| 11.8 Complex Oscillating Synchronization  | 245 |
| 11.9 DTM Induced Oscillating Synchronization: Mackey-Glass & Ikeda Systems              | 245 |
| 11.9.1 Coupled Mackey-Glass Systems   | 245 |
| 11.9.2 Coupled Ikeda Systems  | 246 |
| 11.10 Summary   | 248 |
| References  | 249 |
| <b>12 Exact Solutions of Certain Time Delay Systems: The Car-Following Models</b>       | 251 |
| 12.1 Introduction   | 251 |
| 12.2 The Car-Following Models   | 251 |
| 12.3 The Newell Model   | 252 |
| 12.4 The tanh Car-Following Model   | 255 |
| 12.5 Other Developments   | 257 |
| References  | 258 |

**Appendix**

|   |     |
|---|-----|
| <b>A Computing Lyapunov Exponents for Time-Delay Systems</b> .....                  | 259 |
| A.1 Introduction .....  | 259 |
| A.2 Lyapunov Exponents of an n-Dimensional Dynamical System .....                   | 259 |
| A.2.1 Computation of Lyapunov Exponents .....                                       | 260 |
| A.3 Lyapunov Exponents of a DDE .....   | 261 |
| References .....  | 262 |
| <b>B A Brief Introduction to Synchronization in Chaotic Dynamical Systems</b> ..... | 263 |
| B.1 Introduction .....  | 263 |
| B.2 Characterization of Synchronization .....                                       | 265 |
| B.2.1 Complete Synchronization .....  | 268 |
| B.2.2 Phase Synchronization .....   | 269 |
| B.2.3 Lag Synchronization .....   | 271 |
| B.2.4 Anticipatory Synchronization .....  | 272 |
| B.2.5 Generalized Synchronization .....   | 273 |
| References .....  | 275 |
| <b>C Recurrence Analysis</b> .....  | 279 |
| C.1 Introduction .....  | 279 |
| C.2 Recurrence Plots and Their Variants .....                                       | 280 |
| C.2.1 Recurrence Plots .....  | 280 |
| C.2.2 Cross Recurrence Plots (CRP) .....  | 282 |
| C.2.3 Joint Recurrence Plots (JRP) .....  | 283 |
| C.3 Recurrence Quantification Analysis (RQA) .....                                  | 284 |
| C.3.1 Generalized Autocorrelation Function, $P(t)$ .....                            | 284 |
| C.3.2 Correlation of Probability of Recurrence (CPR) .....                          | 285 |
| C.3.3 Joint Probability of Recurrence (JPR) .....                                   | 285 |
| C.3.4 Similarity of Probability of Recurrence (SPR) .....                           | 286 |
| C.4 Synchronization and Recurrences .....   | 286 |
| C.4.1 PS in Mutually Coupled Rössler Systems .....                                  | 286 |
| C.4.2 Phase to Lag Synchronization .....  | 288 |
| References .....  | 291 |
| <b>D Some More Examples of DDEs</b> .....   | 293 |
| D.1 Introduction .....  | 293 |
| D.2 DDEs with Constant Delay .....  | 293 |
| D.2.1 Hutchinson's Equation/Delayed Logistic Equation .....                         | 293 |
| D.2.2 Gopalsamy and Ladas Population Model .....                                    | 293 |
| D.2.3 Stem-Cell Model .....   | 294 |
| D.2.4 Pupil Cycling Model .....   | 294 |

|   |     |
|---|-----|
| D.3 DDEs with Discrete Delays .....                   | 295 |
| D.3.1 Australian Blowfly Model .....                  | 295 |
| D.3.2 Wilson and Cowan Model .....                    | 295 |
| D.3.3 Human Respiratory Model .....                   | 295 |
| D.4 DDEs with Distributed Delay .....                 | 296 |
| D.4.1 Volterra's Logistic Equation .....              | 296 |
| D.4.2 Neural Network with Distributed Delay .....     | 296 |
| D.4.3 Chemostat Model .....                           | 297 |
| D.5 DDEs with State-Dependent Delay .....             | 297 |
| D.5.1 Population Model .....                          | 297 |
| D.5.2 Logistic Model with State Dependent Delay ..... | 297 |
| D.5.3 Mechanical Model for Machine Tool Chatter ..... | 298 |
| D.6 DDEs with Time-Dependent Delay .....              | 298 |
| D.6.1 Stem-Cell Equation .....                        | 298 |
| D.6.2 Neural Network Model .....                      | 298 |
| References .....                                      | 299 |
| <b>Glossary</b> .....                                 | 301 |
| <b>Index</b> .....                                    | 309 |