Contents

Preface *page* xiii

Part III Flow and dissipation 1

12 Waves and instabilities of stationary plasmas 3
12.1 Laboratory and astrophysical plasmas 3
12.1.1 Grand vision: magnetized plasma on all scales 3
12.1.2 Differences between laboratory and astrophysical plasmas 6
12.1.3 Plasmas with background flow 12
12.2 Spectral theory of stationary plasmas 13
12.2.1 Basic equations 13
12.2.2 Frieman–Rotenberg formulation 16
12.2.3 Self-adjointness of the generalized force operator* 22
12.2.4 Energy conservation and stability 27
12.3 Solution paths in the complex ω plane 35
12.3.1 Opening up the boundaries 35
12.3.2 Approach to eigenvalues 40
12.4 Literature and exercises 47

13 Shear flow and rotation 49
13.1 Spectral theory of plane plasmas with shear flow 49
13.1.1 Gravito-MHD wave equation for plane plasma flow 49
13.1.2 Kelvin–Helmholtz instabilities in interface plasmas 55
13.1.3 Continua and oscillation theorem $\mathcal{R}$ for real eigenvalues 59
13.1.4 Complex eigenvalues and the alternator 65
13.2 Case study: flow-driven instabilities in diffuse plasmas 71
13.2.1 Rayleigh–Taylor instabilities of magnetized plasmas 73
13.2.2 Kelvin–Helmholtz instabilities of ordinary fluids 76
13.2.3 Gravito-MHD instabilities of stationary plasmas 85

13.2.4 Oscillation theorem $\mathcal{C}$ for complex eigenvalues 91
13.3 Spectral theory of rotating plasmas 93
13.3.1 MHD wave equation for cylindrical flow 93
13.3.2 Local stability* 98
13.3.3 WKB approximation 102
13.4 Rotational instabilities 104
13.4.1 Rigid rotation of incompressible plasmas 104
13.4.2 Magneto-rotational instability: local analysis 112
13.4.3 Magneto-rotational instability: numerical solutions 118
13.5 Literature and exercises 123

14 Resistive plasma dynamics 127
14.1 Plasmas with dissipation 127
14.1.1 Conservative versus dissipative dynamical systems 127
14.1.2 Stability of force-free magnetic fields: a trap 128
14.2 Resistive instabilities 135
14.2.1 Basic equations 135
14.2.2 Tearing modes 138
14.2.3 Resistive interchange modes 149
14.3 Resistive spectrum 150
14.3.1 Resistive wall mode 150
14.3.2 Spectrum of homogeneous plasma 155
14.3.3 Spectrum of inhomogeneous plasma 158
14.4 Reconnection 162
14.4.1 Reconnection in 2D Harris sheet 162
14.4.2 Petschek reconnection 168
14.4.3 Kelvin–Helmholtz induced tearing instabilities 169
14.4.4 Extended MHD and reconnection 171
14.5 Literature and exercises 175

15 Computational linear MHD 177
15.1 Spatial discretization techniques 178
15.1.1 Basic concepts for discrete representations 180
15.1.2 Finite difference methods 182
15.1.3 Finite element method 186
15.1.4 Spectral methods 196
15.1.5 Mixed representations 201
15.2 Linear MHD: boundary value problems 204
15.2.1 Linearized MHD equations 204
15.2.2 Steady solutions to linearly driven problems 206
15.2.3 MHD eigenvalue problems 209

15.2.4 Extended MHD examples 211
15.3 Linear algebraic methods 217
15.3.1 Direct and iterative linear system solvers 217
15.3.2 Eigenvalue solvers: the QR algorithm 220
15.3.3 Inverse iteration for eigenvalues and eigenvectors 221
15.3.4 Jacobi–Davidson method 222
15.4 Linear MHD: initial value problems 225
15.4.1 Temporal discretizations: explicit methods 225
15.4.2 Disparateness of MHD time scales 233
15.4.3 Temporal discretizations: implicit methods 234
15.4.4 Applications: linear MHD evolutions 236
15.5 Concluding remarks 240
15.6 Literature and exercises 241

Part IV Toroidal plasmas 245

16 Static equilibrium of toroidal plasmas 247
16.1 Axi-symmetric equilibrium 247
16.1.1 Equilibrium in tokamaks 247
16.1.2 Magnetic field geometry 252
16.1.3 Cylindrical limits 256
16.1.4 Global confinement and parameters 260
16.2 Grad–Shafranov equation 269
16.2.1 Derivation of the Grad–Shafranov equation 269
16.2.2 Large aspect ratio expansion: internal solution 271
16.2.3 Large aspect ratio expansion: external solution 277
16.3 Exact equilibrium solutions 284
16.3.1 Poloidal flux scaling 284
16.3.2 Soloviev equilibrium 289
16.3.3 Numerical equilibria* 293
16.4 Extensions 299
16.4.1 Toroidal rotation 299
16.4.2 Gravitating plasma equilibria* 301
16.4.3 Challenges 302
16.5 Literature and exercises 304

17 Linear dynamics of static toroidal plasmas 307
17.1 "Ad more geometrico" 307
17.1.1 Alfvén wave dynamics in toroidal geometry 307
17.1.2 Coordinates and mapping 308
17.1.3 Geometrical–physical characteristics 309

17.2 Analysis of waves and instabilities in toroidal geometry 315
17.2.1 Spectral wave equation 315
17.2.2 Spectral variational principle 318
17.2.3 Alfvén and slow continuum modes 319
17.2.4 Poloidal mode coupling 322
17.2.5 Alfvén and slow ballooning modes 326
17.3 Computation of waves and instabilities in tokamaks 334
17.3.1 Ideal MHD versus resistive MHD in computations 334
17.3.2 Edge localized modes 340
17.3.3 Internal modes 344
17.3.4 Toroidal Alfvén eigenmodes and MHD spectroscopy 347
17.4 Literature and exercises 352

18 Linear dynamics of stationary toroidal plasmas* 355
18.1 Transonic toroidal plasmas 355
18.2 Axi-symmetric equilibrium of transonic stationary states* 357
18.2.1 General equations and toroidal rescalings* 357
18.2.2 Elliptic and hyperbolic flow regimes* 365
18.2.3 Expansion of the equilibrium in small toroidicity* 366
18.3 Equations for the continuous spectrum* 374
18.3.1 Reduction for straight-field-line coordinates* 374
18.3.2 Continua of poloidally and toroidally rotating plasmas* 378
18.3.3 Analysis of trans-slow continua for small toroidicity* 385
18.4 Trans-slow continua in tokamaks and accretion disks* 392
18.4.1 Tokamaks and magnetically dominated accretion disks* 393
18.4.2 Gravity dominated accretion disks* 396
18.4.3 A new class of transonic instabilities 397
18.5 Literature and exercises* 402

Part V Nonlinear dynamics 405

19 Computational nonlinear MHD 407
19.1 General considerations for nonlinear conservation laws 408
19.1.1 Conservative versus primitive variable formulations 408
19.1.2 Scalar conservation law and the Riemann problem 415
19.1.3 Numerical discretizations for a scalar conservation law 420
19.1.4 Finite volume treatments 430
19.2 Upwind-like finite volume treatments for 1D MHD 433
19.2.1 The Godunov method 434
19.2.2 A robust shock-capturing method: TVDLF 440
19.2.3 Approximate Riemann solver type schemes 446

19.2.4 Simulating 1D MHD Riemann problems 451
19.3 Multi-dimensional MHD computations 454
19.3.1 $\nabla \cdot \mathbf{B} = 0$ condition for shock-capturing schemes 455
19.3.2 Example nonlinear MHD scenarios 461
19.3.3 Alternative numerical methods 466
19.4 Implicit approaches for extended MHD simulations 473
19.4.1 Alternating direction implicit strategies 474
19.4.2 Semi-implicit methods 475
19.4.3 Simulating ideal and resistive instability developments 481
19.4.4 Global simulations for tokamak plasmas 482
19.5 Literature and exercises 484

20 Transonic MHD flows and shocks 487
20.1 Transonic MHD flows 487
20.1.1 Flow in laboratory and astrophysical plasmas 487
20.1.2 Characteristics in space and time 488
20.2 Shock conditions 490
20.2.1 Special case: gas dynamic shocks 492
20.2.2 MHD discontinuities without mass flow 498
20.2.3 MHD discontinuities with mass flow 500
20.2.4 Slow, intermediate and fast shocks 505
20.3 Classification of MHD shocks 507
20.3.1 Distilled shock conditions 507
20.3.2 Time reversal duality 513
20.3.3 Angular dependence of MHD shocks 520
20.3.4 Observational considerations of MHD shocks 527
20.4 Stationary transonic flows 529
20.4.1 Modeling the solar wind–magnetosphere boundary 530
20.4.2 Modeling the solar wind by itself 531
20.4.3 Example astrophysical transonic flows 534
20.5 Literature and exercises 540

21 Ideal MHD in special relativity 543
21.1 Four-dimensional space-time: special relativistic concepts 544
21.1.1 Space-time coordinates and Lorentz transformations 544
21.1.2 Four-vectors in flat space-time and invariants 547
21.1.3 Relativistic gas dynamics and stress–energy tensor 551
21.1.4 Sound waves and shock relations in relativistic gases 556
21.2 Electromagnetism and special relativistic MHD 564
21.2.1 Electromagnetic field tensor and Maxwell's equations 564
21.2.2 Stress–energy tensor for electromagnetic fields 569

21.2.3 Ideal MHD in special relativity 570
21.2.4 Wave dynamics in a homogeneous plasma 572
21.2.5 Shock conditions in relativistic MHD 577
21.3 Computing relativistic magnetized plasma dynamics 580
21.3.1 Numerical challenges from relativistic MHD 583
21.3.2 Example astrophysical applications 584
21.4 Literature and exercises 588

Appendices 591

A Vectors and coordinates 591
A.1 Vector identities 591
A.2 Vector expressions in orthogonal coordinates 592
A.3 Vector expressions in non-orthogonal coordinates 600

References 604
Index 629