
Contents

1	Introduction	1
1.1	Why piecewise smooth?	1
1.2	Impact oscillators	3
1.2.1	Case study I: A one-degree-of-freedom impact oscillator	6
1.2.2	Periodic motion	13
1.2.3	What do we actually see?	18
1.2.4	Case study II: A bilinear oscillator	26
1.3	Other examples of piecewise-smooth systems	28
1.3.1	Case study III: Relay control systems	28
1.3.2	Case study IV: A dry-friction oscillator	32
1.3.3	Case study V: A DC–DC converter	34
1.4	Non-smooth one-dimensional maps	39
1.4.1	Case study VI: A simple model of irregular heartbeats	39
1.4.2	Case study VII: A square-root map	42
1.4.3	Case study VIII: A continuous piecewise-linear map	44
2	Qualitative theory of non-smooth dynamical systems	47
2.1	Smooth dynamical systems	47
2.1.1	Ordinary differential equations (flows)	49
2.1.2	Iterated maps	53
2.1.3	Asymptotic stability	58
2.1.4	Structural stability	59
2.1.5	Periodic orbits and Poincaré maps	63
2.1.6	Bifurcations of smooth systems	67
2.2	Piecewise-smooth dynamical systems	71
2.2.1	Piecewise-smooth maps	71
2.2.2	Piecewise-smooth ODEs	73
2.2.3	Filippov systems	75
2.2.4	Hybrid dynamical systems	78
2.3	Other formalisms for non-smooth systems	83
2.3.1	Complementarity systems	83

2.3.2	Differential inclusions	88
2.3.3	Control systems	91
2.4	Stability and bifurcation of non-smooth systems	93
2.4.1	Asymptotic stability	94
2.4.2	Structural stability and bifurcation	96
2.4.3	Types of discontinuity-induced bifurcations	100
2.5	Discontinuity mappings	103
2.5.1	Transversal intersections; a motivating calculation	105
2.5.2	Transversal intersections; the general case	107
2.5.3	Non-transversal (grazing) intersections	111
2.6	Numerical methods	114
2.6.1	Direct numerical simulation	115
2.6.2	Path-following	118
3	Border-collision in piecewise-linear continuous maps	121
3.1	Locally piecewise-linear continuous maps	121
3.1.1	Definitions	124
3.1.2	Possible dynamical scenarios	125
3.1.3	Border-collision normal form map	127
3.2	Bifurcation of the simplest orbits	128
3.2.1	A general classification theorem	128
3.2.2	Notation for bifurcation classification	131
3.3	Equivalence of border-collision classification methods	137
3.3.1	Observer canonical form	137
3.3.2	Proof of Theorem 3.1	140
3.4	One-dimensional piecewise-linear maps	143
3.4.1	Periodic orbits of the map	145
3.4.2	Bifurcations between higher modes	147
3.4.3	Robust chaos	149
3.5	Two-dimensional piecewise-linear normal form maps	154
3.5.1	Border-collision scenarios	155
3.5.2	Complex bifurcation sequences	157
3.6	Maps that are noninvertible on one side	159
3.6.1	Robust chaos	159
3.6.2	Numerical examples	164
3.7	Effects of nonlinear perturbations	169
4	Bifurcations in general piecewise-smooth maps	171
4.1	Types of piecewise-smooth maps	171
4.2	Piecewise-smooth discontinuous maps	174
4.2.1	The general case	174
4.2.2	One-dimensional discontinuous maps	176
4.2.3	Periodic behavior: $l = -1, \nu_1 > 0, \nu_2 < 1$	180
4.2.4	Chaotic behavior: $l = -1, \nu_1 > 0, 1 < \nu_2 < 2$	185
4.3	Square-root maps	188

4.3.1	The one-dimensional square-root map	188
4.3.2	Quasi one-dimensional behavior	193
4.3.3	Periodic orbits bifurcating from the border-collision	199
4.3.4	Two-dimensional square-root maps	205
4.4	Higher-order piecewise-smooth maps	210
4.4.1	Case I: $\gamma = 2$	211
4.4.2	Case II: $\gamma = 3/2$	213
4.4.3	Period-adding scenarios	214
4.4.4	Location of the saddle-node bifurcations	217
5	Boundary equilibrium bifurcations in flows	219
5.1	Piecewise-smooth continuous flows	219
5.1.1	Classification of simplest BEB scenarios	221
5.1.2	Existence of other attractors	225
5.1.3	Planar piecewise-smooth continuous systems	226
5.1.4	Higher-dimensional systems	229
5.1.5	Global phenomena for persistent boundary equilibria	232
5.2	Filippov flows	233
5.2.1	Classification of the possible cases	235
5.2.2	Planar Filippov systems	237
5.2.3	Some global and non-generic phenomena	242
5.3	Equilibria of impacting hybrid systems	245
5.3.1	Classification of the simplest BEB scenarios	246
5.3.2	The existence of other invariant sets	249
6	Limit cycle bifurcations in impacting systems	253
6.1	The impacting class of hybrid systems	253
6.1.1	Examples	255
6.1.2	Poincaré maps related to hybrid systems	261
6.2	Discontinuity mappings near grazing	265
6.2.1	The geometry near a grazing point	266
6.2.2	Approximate calculation of the discontinuity mappings .	271
6.2.3	Calculating the PDM	271
6.2.4	Approximate calculation of the ZDM	273
6.2.5	Derivation of the ZDM and PDM using Lie derivatives .	274
6.3	Grazing bifurcations of periodic orbits	279
6.3.1	Constructing compound Poincaré maps	280
6.3.2	Unfolding the dynamics of the map	284
6.3.3	Examples	285
6.4	Chattering and the geometry of the grazing manifold	295
6.4.1	Geometry of the stroboscopic map	295
6.4.2	Global behavior of the grazing manifold \mathcal{G}	296
6.4.3	Chattering and the set $G^{(\infty)}$	299
6.5	Multiple collision bifurcation	302

7	Limit cycle bifurcations in piecewise-smooth flows	307
7.1	Definitions and examples	307
7.2	Grazing with a smooth boundary	318
7.2.1	Geometry near a grazing point	319
7.2.2	Discontinuity mappings at grazing	321
7.2.3	Grazing bifurcations of periodic orbits	325
7.2.4	Examples	327
7.2.5	Detailed derivation of the discontinuity mappings	334
7.3	Boundary-intersection crossing bifurcations	340
7.3.1	The discontinuity mapping in the general case	341
7.3.2	Derivation of the discontinuity mapping in the corner-collision case	346
7.3.3	Examples	347
8	Sliding bifurcations in Filippov systems	355
8.1	Four possible cases	355
8.1.1	The geometry of sliding bifurcations	356
8.1.2	Normal form maps for sliding bifurcations	359
8.2	Motivating example: a relay feedback system	364
8.2.1	An adding-sliding route to chaos	366
8.2.2	An adding-sliding bifurcation cascade	368
8.2.3	A grazing-sliding cascade	370
8.3	Derivation of the discontinuity mappings	373
8.3.1	Crossing-sliding bifurcation	375
8.3.2	Grazing-sliding bifurcation	377
8.3.3	Switching-sliding bifurcation	381
8.3.4	Adding-sliding bifurcation	382
8.4	Mapping for a whole period: normal form maps	383
8.4.1	Crossing-sliding bifurcation	384
8.4.2	Grazing-sliding bifurcation	390
8.4.3	Switching-sliding bifurcation	393
8.4.4	Adding-sliding bifurcation	395
8.5	Unfolding the grazing-sliding bifurcation	396
8.5.1	Non-sliding period-one orbits	396
8.5.2	Sliding orbit of period-one	397
8.5.3	Conditions for persistence or a non-smooth fold	399
8.5.4	A dry-friction example	399
8.6	Other cases	403
8.6.1	Grazing-sliding with a repelling sliding region — catastrophe	403
8.6.2	Higher-order sliding	404

9 Further applications and extensions	409
9.1 Experimental impact oscillators: noise and parameter sensitivity	409
9.1.1 Noise	410
9.1.2 An impacting pendulum: experimental grazing bifurcations	412
9.1.3 Parameter uncertainty	419
9.2 Rattling gear teeth: the similarity of impacting and piecewise-smooth systems	422
9.2.1 Equations of motion	423
9.2.2 An illustrative case	425
9.2.3 Using an impacting contact model	426
9.2.4 Using a piecewise-linear contact model	431
9.3 A hydraulic damper: non-smooth invariant tori	434
9.3.1 The model	436
9.3.2 Grazing bifurcations	438
9.3.3 A grazing bifurcation analysis for invariant tori	441
9.4 Two-parameter sliding bifurcations in friction oscillators	448
9.4.1 A degenerate crossing-sliding bifurcation	449
9.4.2 Fold bifurcations of grazing-sliding limit cycles	453
9.4.3 Two simultaneous grazings	455
References	459
Index	475