

Just imagine a feeble knight-errant, armed only with a paper sword – and, perhaps, a laptop with symbolic computation software: can he dare to overcome a horde of nonlinear beasts? There are, essentially, two tricks known from ancient times: separate them or try to surprise them while they are still waking up. The first method is called *scale separation* and is the principal tool in Chaps. 2 and 3, where we study first fronts, and then structures built up by their combinations in reaction–diffusion and related equations. The second method, called *amplitude expansion*, deriving and analyzing universal equations near symmetry-breaking transitions, prevails in Chaps. 4 and 5.

Reaction–diffusion systems, with their plethora of chemical and biological applications and common but versatile structure, provide the bulk of material of the book. Other common pattern-forming systems rooted in fluid mechanics and nonlinear optics are not considered explicitly, but they converge to the same universal equations of amplitude and phase dynamics.

I am indebted to many colleagues and friends of our “nonlinear community”, which has forged during the late decades of 20th century our basic understanding of pattern formation far from equilibrium. The new century, that seems to be interested, in its youth, in the particular more than in the universal, may still find this knowledge useful for building up future biomorphic technologies based on bottom-up self-organization rather than top-down manufacturing.

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