

## Contents

### Volume I

Preface	XXI
Part I	
Boundary Value Problems for the Laplace Operator in Domains Perturbed Near Isolated Singularities	
Chapter 1 Dirichlet and Neumann Problems for the Laplace Operator in Domains with Corners and Cone Vertices	
1.1 Boundary Value Problems for the Laplace Operator in a Strip	3
1.1.1 The Dirichlet problem	3
1.1.2 The complex Fourier transform	5
1.1.3 Asymptotics of solution of the Dirichlet problem	6
1.1.4 The Neumann problem	7
1.1.5 Final remarks	8
1.2 Boundary Value Problems for the Laplace Operator in a Sector	8
1.2.1 Relationship between the boundary value problems in a sector and a strip	8
1.2.2 The Dirichlet problem	10
1.2.3 The Neumann problem	10
1.3 The Dirichlet Problem in a Bounded Domain with Corner	11
1.3.1 Solvability of the boundary value problem	11
1.3.2 Particular solutions of the homogeneous problem	13
1.3.3 Asymptotics of solution	15
1.3.4 A domain with a corner outlet to infinity	17
1.3.5 Asymptotics of the solutions for particular right-hand sides	18
1.3.6 The Dirichlet problem for the operator $A - 1$	22
1.3.7 The Dirichlet problem in a domain with piecewise smooth boundary	26
1.4 The Neumann Problem in a Bounded Domain with a Corner	30
1.5 Boundary Value Problems for the Laplace Operator in a Punctured Domain and the Exterior of a Bounded Planar Domain	34
1.5.1 Dirichlet and Neumann problems in a punctured planar domain	34
1.5.2 Boundary value problems in the exterior of a bounded domain	36

1.6	Boundary Value Problems in Multi-Dimensional Domains	37
1.6.1	A domain with a conical point	37
1.6.2	A punctured domain	39
1.6.3	Boundary value problems in the exterior of a bounded domain	40
Chapter 2 Dirichlet and Neumann Problems in Domains with Singularly Perturbed Boundaries		
2.1	The Dirichlet Problem for the Laplace Operator in a Three-Dimensional Domain with Small Hole	44
2.1.1	Domains and boundary value problems	44
2.1.2	Asymptotics of the solution. The method of compound expansions	45
2.1.3	Asymptotics of the solution. The method of matched expansions	47
2.1.4	Comparison of asymptotic representations	51
2.2	The Dirichlet Problem for the Operator $A - 1$ in a Three-Dimensional Domain with a Small Hole	52
2.3	Mixed Boundary Value Problems for the Laplace Operator in a Three-Dimensional Domain with a Small Hole	54
2.3.1	The boundary value problem with Dirichlet condition at the boundary of the hole	54
2.3.2	First version of the construction of asymptotics	55
2.3.3	Second version of the construction of asymptotics	57
2.3.4	The boundary value problem with the Neumann condition at the boundary of the gap	59
2.4	Boundary Value Problems for the Laplace Operator in a Planar Domain with a Small Hole	59
2.4.1	Dirichlet problem	60
2.4.2	Mixed boundary value problems	64
2.5	The Dirichlet Problem for the Operator $A - 1$ in a Domain Perturbed Near a Vertex	67
2.5.1	Formulation of the problem	67
2.5.2	The first terms of the asymptotics	67
2.5.3	Admissible series	70
2.5.4	Redistribution of discrepancies	71
2.5.5	The set of exponents in the powers of $\epsilon$ , $r$ , and $p$	72

Part II

General Elliptic Boundary Value Problems in Domains Perturbed  
Near Isolated Singularities of the Boundary

Chapter 3 Elliptic Boundary Value Problems in Domains with Smooth  
Boundaries, in a Cylinder, and in Domains with Cone Vertices

3.1 Boundary Value Problems in Domains with Smooth Boundaries	79
3.1.1 The operator of an elliptic boundary value problem	79
3.1.2 Elliptic boundary value problems in Sobolev and Holder spaces	80
3.1.3 The adjoint boundary value problem (the case of normal boundary conditions)	83
3.1.4 Adjoint operator in spaces of distributions	84
3.1.5 Elliptic boundary value problems depending on a complex parameter	85
3.1.6 Boundary value problems for elliptic systems	89
3.2 Boundary value problems in cylinders and cones	92
3.2.1 Solvability of boundary value problems in cylinders: the case of coefficients independent of $t$	92
3.2.2 Asymptotics at infinity of solutions to boundary value problems in cylinders with coefficients independent of $t$	95
3.2.3 Solvability of boundary value problems in a cone	97
3.2.4 Asymptotics of the solutions at infinity and near the vertex of a cone for boundary value problems with coefficients independent of $r$	99
3.2.5 Boundary value problems for elliptic systems in a cone	100
3.2.6 Asymptotics of the solution for the right-hand side given by an asymptotic expansion	104
3.3 Boundary Value Problems in Domains with Cone Vertices	106
3.3.1 Statement of the problem	106
3.3.2 Asymptotics of the solution near a cone vertex	107
3.3.3 Formulas for coefficients in the asymptotics of solution (under simplified assumptions)	109
3.3.4 Formula for coefficients in the asymptotics of solution (general case)	110
3.3.5 Index of the boundary value problem	113

Chapter 4 Asymptotics of Solutions to General Elliptic Boundary Value

## Problems in Domains Perturbed Near Cone Vertices

4.1 Formulation of the Boundary Value Problems and some Preliminary Considerations	115
4.1.1 The domains	115
4.1.2 Admissible scalar differential operators	116
4.1.3 Limit operators	117
4.1.4 Matrices of differential operators	118
4.1.5 Boundary value problems	118
4.1.6 Function spaces with norms depending on the parameter $\epsilon$	118
4.2 Transformation of the Perturbed Boundary Value Problem into a System of Equations and a Theorem about the Index	120
4.2.1 The limit operator	120
4.2.2 Reduction of the problem to a system	121
4.2.3 Reconstruction of the original problem from the system	124
4.2.4 Fredholm property for the operator of the boundary value problem in a domain with singularly perturbed boundary	127
4.2.5 On the index of the original problem	127
4.3 Asymptotic Expansions of Data in the Boundary Value Problem	130
4.3.1 Asymptotic expansion of the coefficients and the right-hand sides	131
4.3.2 Asymptotic formulas for solutions of the limit problems	132
4.3.3 Asymptotic expansions of operators of the boundary value problem	133
4.3.4 Preliminary description of algorithm for construction of the asymptotics of solutions	134
4.3.5 The set of exponents in asymptotics of solutions of the limit problems	137
4.3.6 Formal expansion for the operator in powers of small parameter	138
4.4 Construction and Justification of the Asymptotics of Solution of the Boundary Value Problem	140
4.4.1 The problem in matrix notation	140
4.4.2 Auxiliary operators and their properties	141
4.4.3 Formal asymptotics of the solution in the case of uniquely solvable limit problems	142
4.4.4 A particular basis in the cokernel of the operator $M_0$	144

4.4.5 Formal solution in the case of non-unique solvability of the limit problems	149
4.4.6 Asymptotics of the solution of the singularly perturbed problem	154
Chapter 5 Variants and Corollaries of the Asymptotic Theory	
5.1 Estimates of Solutions of the Dirichlet Problem for the Helmholtz Operator in a Domain with Boundary Smoothened Near a Corner	157
Sobolev Boundary Value Problems	161
General Boundary Value Problem in a Domain with Small Holes	167
Problems with Non-Smooth and Parameter Dependent Data	173
5.4.1 The case of a non-smooth domain	173
5.4.2 The case of parameter dependent auxiliary problems	175
5.4.3 The case of a parameter independent domain	177
Non-Local Perturbation of a Domain with Cone Vertices	182
5.5.1 Perturbations of a domain with smooth boundary	182
5.5.2 Regular perturbation of a domain with a corner	184
5.5.3 A non-local singular perturbation of a planar domain with a corner	186
5.6 Asymptotics of Solutions to Boundary Value Problems in Long Tubular Domains	189
5.6.1 The problem	189
5.6.2 Limit problems	190
5.6.3 Solvability of the original problem	192
5.6.4 Expansion of the right-hand sides and the set of exponents in the asymptotics	193
5.6.5 Redistribution of defects	195
5.6.6 Coefficients in the asymptotic series	197
5.6.7 Estimate of the remainder term	198
5.6.8 Example	200
5.7 Asymptotics of Solutions of a Quasi-Linear Equation in a Domain with Singularly Perturbed Boundary	201
5.7.1 A three-dimensional domain with a small gap	202
5.7.2 A planar domain with a small gap	207
5.7.3 A domain smoothened near a corner point	213
5.8 Bending of an Almost Polygonal Plate with Freely Supported Boundary	217

5.8.1	Boundary value problems in domains with corners	219
5.8.2	A singularly perturbed domain and limit problems	220
5.8.3	The principal term in the asymptotics	221
5.8.4	The principal term in the asymptotics (continued)	223
Part III		
Asymptotic Behaviour of Functionals on Solutions of Boundary Value Problems in Domains Perturbed Near Isolated Boundary Singularities		
Chapter 6 Asymptotic Behaviour of Intensity Factors for Vertices of Corners and Cones Coming Close		
6.1	Dirichlet's Problem for Laplace's Operator	228
6.1.1	Statement of the problem	228
6.1.2	Asymptotic behaviour of the coefficient $C_+$	229
6.1.3	Justification of the asymptotic formula for the coefficient $C_+$	230
6.1.4	The case $\epsilon \rightarrow 0$	231
6.1.5	The two-dimensional case	231
6.2	Neumann's Problem for Laplace's Operator	232
6.2.1	Statement of the problem	232
6.2.2	Boundary value problems	232
6.2.3	The case of disconnected boundary	234
6.2.4	The case of connected boundary	235
6.3	Intensity Factors for Bending of a Thin Plate with a Crack	235
6.3.1	Statement of the problem	235
6.3.2	Clamped cracks (The asymptotic behaviour near crack tips)	236
6.3.3	Fixedly clamped cracks (Asymptotic behaviour of the intensity factors)	237
6.3.4	Freely supported cracks	238
6.3.5	Free cracks (The asymptotic behaviour of solution near crack vertices)	240
6.3.6	Free cracks (The asymptotic behaviour of intensity factors)	240
6.4	Antiplanar and Planar Deformations of Domains with Cracks	243
6.4.1	Torsion of a bar with a longitudinal crack	243
6.4.2	The two-dimensional problem of the elasticity theory in a domain with collinear close cracks	245

Chapter 7 Asymptotic Behaviour of Energy Integrals for Small Perturbations of the Boundary Near Corners and Isolated Points	
7.1 Asymptotic Behaviour of Solutions of the Perturbed Problem	251
7.1.1 The unperturbed boundary value problem	251
7.1.2 Perturbed problem	254
7.1.3 The second limit problem	254
7.1.4 Asymptotic behaviour of solutions of the perturbed problem	256
7.1.5 The case of right-hand sides localized near a point	259
7.2 Asymptotic Behaviour of a Bilinear Form	261
7.2.1 The asymptotic behaviour of a bilinear form (the general case)	261
7.2.2 Asymptotic behaviour of a bilinear form for right-hand sides localized near a point	264
7.2.3 Asymptotic behaviour of a quadratic form	266
7.3 Asymptotic Behaviour of a Quadratic Form for Problems in Regions with Small Holes	267
7.3.1 Statement of the problem	267
7.3.2 The case of uniquely solvable boundary problems	267
7.3.3 The case of the critical dimension	270
Chapter 8 Asymptotic Behaviour of Energy Integrals for Particular Problems of Mathematical Physics	
8.1 Dirichlet's Problem for Laplace's Operator	277
8.1.1 Perturbation of a domain near a corner or conic point	277
8.1.2 The case of right-hand sides depending on $\mu$	280
8.1.3 The case of right-hand sides depending on $x$ and $\mu$	281
8.1.4 Dirichlet's problem for Laplace's operator in a domain with a small hole	282
8.1.5 Refinement of the asymptotic behaviour	284
8.1.6 Two-dimensional domains with a small hole	287
8.1.7 Dirichlet's problem for Laplace's operator in domains with several small holes	288
8.2 Neumann's Problem in Domains with one Small Hole	291
8.3 Dirichlet's Problem for the Biharmonic Equation in a Domain	

with Small Holes	293
8.4 Variation of Energy Depending on the Length of Crack	296
8.4.1 The antiplanar deformation	296
8.4.2 A problem in the two-dimensional elasticity	299
8.5 Remarks on the Behaviour of Solutions of Problems in the Two-dimensional Elasticity Near Corner Points	302
8.5.1 Statement of problems	302
8.5.2 The asymptotic behaviour of solutions of the antiplanar deformation problem	302
8.5.3 Asymptotic behaviour of solutions of the planar deformation problem	303
8.5.4 Boundary value problems in unbounded domains	306
8.6 Derivation of Asymptotic Formulas for Energy	308
8.6.1 Statement of problems	308
8.6.2 Antiplanar deformation	309
8.6.3 Planar deformation	310
8.6.4 Refinement of the asymptotic formula for energy	311
8.6.5 Defect in the material near vertex of the crack	313

## Part IV

### Asymptotic Behaviour of Eigenvalues of Boundary Value Problems in Domains with Small Holes

#### Chapter 9 Asymptotic Expansions of Eigenvalues of Classic Boundary Value Problems

9.1 Asymptotic Behaviour of the First Eigenvalue of a Mixed Boundary Value Problem	318
9.1.1 Statement of the problem	318
9.1.2 The three-dimensional case (formal asymptotic representation)	319
9.1.3 The planar case (formal asymptotic representation)	322
9.1.4 Justification of asymptotic expansions in the three-dimensional case	326
9.1.5 Justification of asymptotic expansions in the two-dimensional case	329
9.2 Asymptotic Expansions of Eigenvalues of Other Boundary Value Problems	331



9.2.1 Dirichlet's problem in a three-dimensional domain with a small hole	331
9.2.2 Mixed boundary value problem in domains with several small holes	334
9.2.3 Mixed boundary value problem with Neumann's condition on the boundary of small hole	337
9.2.4 Dirichlet's problem on a Riemannian manifold with a small hole	340
XII	
Contents	
9.3 Asymptotic Representations of Eigenvalues of Problems of the Elasticity Theory for Bodies with Small Inclusions and Holes	342
9.3.1 Statement of the problem	342
9.3.2 Structure of the asymptotic representation	343
9.3.3 Particular solutions of the boundary layer problem	343
9.3.4 Perturbation of the eigenvalue $\Lambda_0$	347
9.3.5 Problem in the two-dimensional elasticity (one hole with a free surface)	349
Chapter 10 Homogeneous Solutions of Boundary Value Problems in the Exterior of a Thin Cone	
10.1 Formal Asymptotic Representation	355
10.1.1 Statement of the problem	355
10.1.2 The case $n-1 > 2m$	356
10.1.3 The case $n-1 = 2m$	360
10.2 Inversion of the Principal Part of an Operator Pencil on the Unit Sphere with a Small Hole. An Auxiliary Problem with Matrix Operator	363
10.2.1 "Nearly inverse" operator (the case $2m < n-1$ )	363
10.2.2 "Nearly inverse" operator (the case $2m = n-1$ )	367
10.2.3 Reduction to a problem with a matrix operator (the case $2m < n-1$ )	371
10.2.4 Reduction to a problem with a matrix operator (the case $2m = n-1$ )	373
10.3 Justification of the Asymptotic Behaviour of Eigenvalues (The Case $2m < n-1$ )	374
10.4 Justification of the Asymptotic Behaviour of Eigenvalues (The Case $2m = n-1$ )	379

10.5 Examples and Corollaries	388
10.5.1 A scalar operator	388
10.5.2 Lamé's and Stokes' systems	389
10.5.3 Continuity at the cone vertex of solution of Dirichlet's problem	390
10.6 Examples of Discontinuous Solutions to Dirichlet's Problem in Domains with a Conic Point	391
10.6.1 Equation of second order with discontinuous solutions	391
10.6.2 Dirichlet's problem for an elliptic equation of the fourth order with real coefficients	393
10.7 Singularities of Solutions of Neumann's Problem	395
10.7.1 Introduction	395
10.7.2 Formal asymptotic representation	396
10.8 Justification of the Asymptotic Formulas	400
10.8.1 Multiplicity of the spectrum near the point $A = 2$	400
10.8.2 Nearly inverse operator for Neumann's problem in $G_J$	401
10.8.3 Justification of asymptotic representation of eigenvalues	406
Contents	
XIII	
Comments on Parts I-IV	
Comments on Part I	411
Chapter 1	411
Chapter 2	411
Comments on Part II	411
Chapter 3	411
Chapter 4	412
Chapter 5	412
Comments on Part III	412
Chapter 6	412
Chapter 7	408
Chapter 8	412
Comments on Part IV	412
Chapter 9	412
Chapter 10	412
List of Symbols	413
1. Basic Symbols	413
2. Symbols for function spaces and related concepts	414

3. Symbols for functions, distributions and related concepts	415
4. Other symbols	415
References	417
Index	433