

Contents

1. Reaction-Diffusion Equations	1
1.1 Introduction	1
1.2 Bifurcations and Pattern Formations	2
1.3 Boundary Conditions	4
2. Continuation Methods	7
2.1 Parameterization of Solution Curves	8
2.1.1 Natural parameterization	8
2.1.2 Parameterization with arclength	9
2.1.3 Parameterization with pseudo-arclength	11
2.2 Local Parameterization of Solution Manifolds	14
2.3 Predictor-Corrector Methods	16
2.3.1 Euler-Newton method	19
2.3.2 A continuation-Lanczos algorithm	22
2.3.3 A continuation-Arnoldi algorithm	25
2.4 Computation of Multi-Dimensional Solution Manifolds	27
3. Detecting and Computing Bifurcation Points	31
3.1 Generic Bifurcation Points	31
3.1.1 One-parameter problems	32
3.1.2 Two-parameter problems	34
3.2 Test Functions	35
3.2.1 Test functions for turning points	36
3.2.2 Test functions for simple bifurcation point	40
3.2.3 Test functions for Hopf bifurcations	43
3.2.4 Minimally extended systems	46
3.3 Computing Simple Bifurcation Points	47
3.3.1 Simple bifurcation points	48
3.3.2 Extended systems	49
3.3.3 Newton-like methods	53
3.3.4 Rank-1 corrections for sparse problems	56
3.3.5 A numerical example	59
3.4 Computing Hopf Bifurcation Points	60
3.4.1 Hopf points	60

3.4.2	Extended systems	62
3.4.3	Newton method for extended systems	67
4.	Branch Switching at Simple Bifurcation Points	69
4.1	Structure of Bifurcating Solution Branches.....	70
4.2	Behavior of the Linearized Operator	73
4.3	Euler-Newton Continuation	75
4.4	Branch Switching via Regularized Systems.....	80
4.5	Other Branch Switching Techniques.....	84
5.	Bifurcation Problems with Symmetry	85
5.1	Basic Group Concepts	86
5.2	Equivariant Bifurcation Problems	90
5.3	Equivariant Branching Lemma	92
5.4	A Semi-linear Elliptic PDE on the Unite Square.....	97
6.	Liapunov-Schmidt Method.....	101
6.1	Liapunov-Schmidt Reduction	101
6.2	Equivariance of the Reduced Bifurcation Equations	104
6.3	Derivatives and Taylor Expansion	105
6.4	Equivalence, Determinacy and Stability	107
6.5	Simple Bifurcation Points	109
6.6	Truncated Liapunov-Schmidt Method	110
6.7	Branch Switching at Multiple Bifurcation Points	112
6.7.1	Branch switching with prescribed tangents.....	113
6.7.2	Branch switching with scaling techniques	114
6.8	Corank-2 Problems with D_m -symmetry	118
6.8.1	Semilinear elliptic PDEs on a square	118
6.8.2	A semilinear elliptic PDE on a hexagon	123
7.	Center Manifold Theory	129
7.1	Center Manifolds and Their Properties	129
7.2	Approximation of Center Manifolds	132
7.3	Liapunov-Schmidt Reduction	136
7.4	Symmetry and Normal Form.....	139
7.4.1	Simple bifurcation points	140
7.4.2	Hopf bifurcations	143
7.5	Waves in Reaction-Diffusion Equations	145
7.5.1	Oscillating waves	148
7.5.2	Long waves	148
7.5.3	Long time and large spatial behavior.....	150

8. A Bifurcation Function for Homoclinic Orbits	151
8.1 A Bifurcation Function	152
8.2 Approximation of Homoclinic Orbits	154
8.3 Solving the Adjoint Variational Problem	156
8.3.1 Preserving the inner product	159
8.3.2 Systems with continuous symmetries	162
8.4 The Approximate Bifurcation Function	163
8.5 Examples	165
8.5.1 Freire <i>et al.</i> 's circuit	165
8.5.2 Kuramoto-Sivashinsky equation	167
9. One-Dimensional Reaction-Diffusion Equations	173
9.1 Introduction	173
9.2 Linear Stability Analysis	175
9.2.1 The general system	175
9.2.2 The Brusselator equations	178
9.3 Solution Branches at Double Bifurcations	180
9.3.1 The reflection symmetry and its induced action	182
9.3.2 $(k, m) = (\text{odd}, \text{odd})$ or $(\text{odd}, \text{even})$	182
9.3.3 $(k, m) = (\text{even}, \text{even})$	184
9.3.4 The Brusselator equations	186
9.4 Central Difference Approximations	187
9.4.1 General systems	187
9.4.2 The Brusselator equations	191
9.5 Numerical Results for the Brusselator Equations	193
9.5.1 The length $\ell = 1$, diffusion rates $d_1 = 1$, $d_2 = 2$	193
9.5.2 The length $\ell = 10$, diffusion rates $d_1 = 1$, $d_2 = 2$	197
10. Reaction-Diffusion Equations on a Square	199
10.1 D_4 -Symmetry	200
10.2 Eigenpairs of the Laplacian	202
10.3 Linear Stability Analysis	204
10.4 Bifurcation Points	207
10.4.1 Steady state bifurcation points	208
10.4.2 Hopf bifurcation points	213
10.5 Mode Interactions	213
10.5.1 Steady/steady state mode interactions	213
10.5.2 Hopf/steady state mode interactions	216
10.5.3 Hopf/Hopf mode interactions	217
10.6 Kernels of $D_u G_0$ and $(D_u G_0)^*$	217
10.7 Liapunov-Schmidt Reduction	221
10.8 Simple and Double Bifurcations	222
10.8.1 Simple bifurcations	222
10.8.2 Double bifurcations induced by the D_4 -symmetries ...	223

11. Normal Forms for Hopf Bifurcations	231
11.1 Introduction	231
11.2 Domain Symmetries and Their Extensions	233
11.3 Actions of D_4 on the Center Eigenspace	235
11.4 The Normal Form	237
11.5 Analysis of the Normal Form	238
11.5.1 Odd parity	239
11.5.2 Even parity	240
11.6 Brusselator Equations	244
11.6.1 Linear stability analysis	245
11.6.2 Bifurcation scenario	247
11.6.3 Nonlinear degeneracy	251
12. Steady/Steady State Mode Interactions	255
12.1 Induced Actions	255
12.2 Interaction of Two D_4 -Modes	258
12.2.1 Interaction of two even modes	258
12.2.2 Interaction of an even mode with an odd mode	260
12.2.3 Interaction of two odd modes	262
12.3 Mode Interactions of Three Modes	263
12.3.1 Induced actions	264
12.3.2 Interactions of the modes $(m, n, k) = (\text{even}, \text{odd}, \text{odd})$..	265
12.3.3 Interactions of the modes $(m, n, k) = (\text{even}, \text{odd}, \text{even})$..	268
12.4 Interactions of Four Modes	269
12.4.1 Interactions of the modes $(m, n, k, l) = (\text{even}, \text{odd},$ even, odd)	271
12.4.2 Interactions of the modes $(m, n, k, l) = (\text{even}, \text{even},$ even, odd)	272
12.5 Reactions with Z_2 -Symmetry	275
13. Hopf/Steady State Mode Interactions	283
13.1 Hopf/Steady State Mode Interactions	283
13.2 Induced Actions	286
13.3 Normal Forms	289
13.4 Bifurcation Scenario	293
13.5 Calculations of the Normal Form	299
14. Homotopy of Boundary Conditions	305
14.1 Boundary Conditions	305
14.1.1 Homotopy of boundary conditions	306
14.1.2 Boundary conditions for different components	307
14.1.3 Mixed boundary conditions along the sides	309
14.1.4 Dynamical boundary conditions	309
14.2 A Brief Review of Sturm-Liouville Theory	309
14.3 Laplacian with Robin Boundary Conditions	312

14.4	Variational Form	316
14.5	Continuity of Solutions along the Homotopy	318
14.6	Neumann and Dirichlet Problems	320
14.7	Properties of Eigenvalues	322
14.7.1	One-dimensional problems	323
14.7.2	Two-dimensional problems	327
15.	Bifurcations along a Homotopy of BCs	331
15.1	Introduction	332
15.2	Stability and Symmetries	333
15.3	Normal Forms	335
15.4	Variations of Bifurcations along the Homotopy	337
15.4.1	$(\kappa_1, \kappa_2) = (\text{odd}, \text{even})$ or $(\text{even}, \text{odd})$	338
15.4.2	$(\kappa_1, \kappa_2) = (\text{odd}, \text{odd})$	339
15.4.3	$(\kappa_1, \kappa_2) = (\text{even}, \text{even})$	340
15.5	A Numerical Example	340
15.5.1	Discretization with finite difference methods	341
15.5.2	Homotopy of $(\kappa_1(\mu), \kappa_2(\mu))$ from (1,2) to (2,3)	345
15.5.3	Homotopy of $(\kappa_1(\mu), \kappa_2(\mu))$ from (1,3) to (2,4)	345
15.5.4	Homotopy of $(\kappa_1(\mu), \kappa_2(\mu))$ from (2,4) to (3,5)	347
15.6	Forced Symmetry-Breaking in BCs	349
15.6.1	Bifurcation points	351
15.6.2	Bifurcation scenarios	354
16.	A Mode Interaction on a Homotopy of BCs	361
16.1	Introduction	361
16.2	Symmetries and Normal Forms	363
16.3	Generic Bifurcation Behavior	365
16.3.1	Solutions with the modes ϕ_1, ϕ_2	366
16.3.2	Pure ϕ_3 -mode solutions	367
16.3.3	Interactions of three modes	367
16.4	Scales of Solution Branches	368
16.5	Secondary Bifurcations	370
16.5.1	Secondary Hopf bifurcations	372
16.6	Truncated Bifurcation Equations	373
16.6.1	Derivatives with respect to homotopy parameter	376
16.7	Reduced Stability	378
16.7.1	Stability of solution branches at $(0, \lambda_1(\mu), \mu)$	379
16.7.2	Stability of solution branches at $(0, \lambda_2(\mu), \mu)$	380
16.7.3	Stability of solution branches at mode interaction	380
16.8	A Numerical Example	381
16.8.1	Solution branches along $(0, \lambda_1(\mu), \mu)$	381
16.8.2	Solution branches along $(0, \lambda_2(\mu), \mu)$	382
16.8.3	Mode interaction	383
16.8.4	Switching and continuation of solution branches	385

List of Figures	389
List of Tables	393
Bibliography	395
Index	411

