Volume 2

Glossary of Abbreviations xv

Preface and Introduction xvii

Preface xvii
Introduction xx
Incompressible Flow xxii
The Finite Element Method xxv
Incompressible Flow and the Finite Element Method xxvi
Overview of this Volume xxxi
Some Subjective Discussion xxxv
Why Finite Elements? Why Not Finite Volumes? xxxvi

3 The Navier–Stokes Equations 447

3.1 Notational Introduction 447
3.2 The Continuum Equations (The PDE's) 450
3.3 Alternate Forms of the Viscous Term 452
3.3.1 Stress-Divergence Form 453
3.3.2 Div-Curl Form 453
3.3.3 Curl Form 454
3.4 Alternate Forms of the Non-Linear Term 454
3.4.1 Divergence Form 454
3.4.2 Rotational Form 454
3.4.3 Skew-Symmetric Form 455
3.4.4 A Symmetric Form 455
3.5 Derived Equations 457
3.5.1 The Pressure Poisson Equation (PPE) 457
3.5.2 The Vorticity Transport Equation (VTE) 458
3.5.3 The Penalized Momentum Equation 459
3.6 Alternate Statements of the NS Equations 461
3.6.1 Velocity-Pressure in Divergence Form 461
3.6.2 Velocity-Pressure in Rotational Form 461
3.6.3 PPE Form 461
3.6.4 The Stream Function-Vorticity ($\psi - \omega$) Formulation 462
3.6.5 The Velocity-Vorticity Formulation 462
3.6.6 Other Formulations 463
3.7 Special Cases of Interest 463
3.7.1 Stokes Flow 463
3.7.2 Inviscid Flow 466
3.7.3 Potential Flow 468
3.7.4 Axisymmetric Flow 469
3.8 Boundary Conditions 470
3.8.1 u–P Equations 471
a. Traction 471
b. Mixed 472
c. Total momentum flux 473
d. Symmetry 473
e. Robin 474
f. OBC's 475
g. More OBC's 480

h. Penalty method OBC's 481
i. Ill-posed OBC's 481
3.8.2 The Pressure Poisson Equation and Pressure Boundary Conditions 482
3.8.3 The Vorticity Transport Equation and Boundary Conditions on the Vorticity 485
a. The 2D stream function-vorticity formulation 485
b. The 3D velocity-vorticity formulation 487
3.9 Initial Conditions (and Well-Posedness) 487
3.9.1 The u–*P* Formulation 487
3.9.2 The PPE Formulation 490
3.9.3 Vorticity-Based Methods 491
3.10 Interim Summary 493
3.10.1 A Well-Posed IBVP for Incompressible Flow, and the Equivalence 'Theorem' 493
3.10.2 Some Ill-Posed Problems 496
3.10.3 The Simplified PPE is also Ill-Posed 497
3.10.4 Fixing the SPPE, and the PPE Paradox 499
3.10.5 PPE Solutions that are not NSE Solutions 500
3.10.6 A Remark on the Penalty Method 502
3.10.7 Key Features of Incompressible Flow 502
3.11 Global Conservation Laws 502
3.11.1 Conservation of Mass 503
3.11.2 Momentum Conservation 503
3.11.3 Kinetic Energy Conservation 503
3.11.4 Vorticity Conservation 505
3.11.5 Enstrophy Conservation 507
3.12 Weak Forms of the PDE's/Natural Boundary Conditions (NBC's) 508
3.12.1 The Conventional **u**–*P* Formulation and the Stress-Divergence Form Combined 508
3.12.2 Other **u**–*P* Formulations 516
a. Full divergence form 516
b. Skew-symmetric form 517
c. Rotational form and other curl forms 517
d. Other recent formulations 520
e. Divergence-free basis functions 521
3.12.3 Pressure Poisson Equation Formulations 522
3.12.4 The Stream Function-Vorticity Formulation 523
3.12.5 Some Ill-posed Formulations 525
3.13 The Finite Element Equations/Discretization of the Weak Form 528
3.13.1 Detailed derivation of one **u**–*P* formulation 529
a. Continuum formulation 529
b. GFEM equations 530
c. Matrix-vector representation 534
d. Ill-posed equations 539
e. Normal and tangential BC's 540
f. Axisymmetric case 545
g. Fixing ill-posed Dirichlet BC's 546
3.13.2 The Choice of Elements 547
a. Introduction and summary tables 547
*b. Null spaces and their effects; pressure modes 559
*c. LBB-stability/div-stability 593

d. Bringing LBB to the rest of the people 600
e. Penalty methods 613
f. Some 2D vs 3D considerations 623
3.13.3 Stabilization [D. J. Silvester] 623
a. Stable vs. stabilized methods 624
b. Equal order interpolation via stabilization 625
c. Stabilized approximations using discontinuous pressure 631
d. Impact on iterative solvers 636
e. Recommendations 639
3.13.4 The Discrete Pressure Poisson Equations (PPE's) 640
a. The consistent PPE 640
b. Some inconsistent (approximate) PPE's 642
3.13.5 Additional Detailed Discussion of the Slightly UNSTABLE but Highly USABLE Q_1Q_0 Element 644
a. Introduction 644
b. General problem statement 644
c. Interior momentum equation 647
d. Interior PPE 651
e. Boundary momentum equations/ NBC's 654
f. The PPE at boundaries 660
g. Flow past a flat plate 671
h. Flow past a corner 673
i. Div $\mathbf{u} = 0$ as a PPE BC 676
j. Q_1Q_0 convergence proof 677
k. Quantitative description of some unstable modes 686
l. The boundary vector, g 691
3.13.6 Higher-Order Elements 694
a. Q_2Q_1 695
b. Q_2Q_{-1} 697
c. Q_2P_{-1} 698
3.13.7 Divergence-Free Elements (and Methods) 699
3.13.8 Conservation Laws Revisited 704
3.13.9 Periodic Boundary Conditions 707
3.14 A Control Volume Finite Element Method 712
*3.15 Variational Principles for Potential and Stokes Flow 716
3.15.1 Introduction 716
3.15.2 Discrete Stokes 717
3.15.3 Discrete Potential 721
3.15.4 Continuous Potential 722
3.15.5 Continuous Stokes 726
3.16 Solution Methods for the Semi-Discretized Time-Dependent (and Steady) Equations 729
3.16.1 Some Time-Integration Methods for the DAE's 733
a. Primitive equations/index 2 735
b. PPE methods/index 1 741
c. Error analysis for index 1 and 2 745
d. Some numerical results (Taylor vortex) 750

*3.16.2 A Model DAE Problem 767
a. Introduction 767
b. Index 2 768
c. Index 1 775
d. Index 0 777
e. Penalty 777
f. Energetics 781
g. Numerical integration (analytical) 782
h. Final exercise 790
*3.16.3 Analytical Solution of the Stokes Equations 791
a. Introduction 791
b. Index 2 791
c. Index 1 794
d. Linear stability theory 796
3.16.4 Three Variable-Step Implicit (Index 2) Methods—and Some Steady-State Methods 797
a. Introduction 797
b. Trapezoid rule 797
c. Backward Euler (BDF1) 803
d. BDF2 805
e. Discussion 807
f. Penalty method 808
3.16.5 An Explicit (Index 1) Method, Plus a Few Tricks 812
3.16.6 Semi-Implicit Projection Methods 824
a. Introduction 824
b. Derivation of an 'optimal' projection method, simplifications thereto, and analysis thereof 826
c. A GFEM (almost) implementation of the second-order projection method—projection 2 849
d. A sampling of projection methods used by others 860
3.16.7 Fully-Implicit Segregated Solution Methods—Transient and Steady-State 863
3.16.8 A Fractional-Step (Index 2) Method 868
3.16.9 Other Methods (Used by Others) 872
a. Methods based on trajectories/characteristics 872
b. Methods based on least squares (LSFEM) 873
c. Methods based on Galerkin least squares 875
3.16.10 A Strategy for Hastening Steady Solutions 875
3.17 Aliasing and Aliasing Instability, Linear and Non-Linear 876
*3.18 A New Look at Two Old Finite Difference Methods 880
3.19 Numerical Example—Impulsive Start 884
3.19.1 Introduction 884
3.19.2 Domain, Mesh, BC's, IC's 886
3.19.3 Two Steady-state Results ($\nu = 0,\ \infty$) 887
3.19.4 Pressure Impulse 895

3.19.5 Minimum Time of Believability; Re = 1000 Results 896
3.19.6 Transient Stokes Flow 899
3.19.7 Divergence for $h \to 0$ 905
3.19.8 A Better Model 911
3.19.9 Drag Coefficients 912
3.19.10 A New Analytical Model 916
3.19.11 A Better Mesh 923
3.19.12 Δt vs. t 928
3.19.13 A Deficient Mesh Design 930
3.19.14 Concluding Remarks 932
3.20 Closure: Some Additional Remarks on the Pressure 934

4 Derived Quantities 937

4.1 Introduction 937
4.2 Two Dimensions 938
4.2.1 Smoothing in General 938
4.2.2 Vorticity 939
4.2.3 Stream Function 941
4.2.4 Heat Flux 943
4.2.5 Forces and Moments 949
4.2.6 A Recommended Method for Computing First Derivatives at Nodes 952
4.2.7 Particle Paths 955
4.2.8 Effective Peclet (Reynolds) Number 957
4.2.9 Pressure Smoothing and Node Moving for Q_1Q_0 958
4.3 Three Dimensions 961
4.3.1 Vorticity 961
4.3.2 Helicity Density 961

Appendix 4 Some More Element Matrices 963

A.4.1 Navier–Stokes; Additional Matrices 963
A.4.1.1 Gradient Matrix 963
A.4.1.2 Divergence Matrix 964
A.4.1.3 Consistent Laplacian Matrix for Q_2Q_1 and Lumped Mass 965

Appendix 5 Vector Projections, Orthogonal and Not—and Projection Methods 967

A.5.1 Introduction 967
A.5.2 The $\wp_{J_0}$-Projection 967
A.5.3 The $\wp^h_{J_0}$-Projection 973
A.5.4 The $\wp_{J_1}$-Projection 978
A.5.5 The $\wp^h_{J_1}$-Projection 982

A.5.6 Sequential Projections 984
A.5.7 The Projection Method 985
A.5.8 Ranking Elements via Projections 987

References **989**

Author Index **Ai-1**

Subject Index **Si-1**