

# Contents<sup>†</sup>

## Volume 1

Preface	xv
Glossary of Abbreviations	xix
<b>1 Introduction</b>	<b>1</b>
1.1 Introduction	1
1.2 Incompressible Flow	3
1.3 The Finite Element Method	6
1.4 Incompressible Flow and the Finite Element Method	11
1.5 Overview of this Volume	12
1.6 Some Subjective Discussion	16
1.7 Why Finite Elements? Why Not Finite Volumes?	17
<b>2 The Advection-Diffusion Equation</b>	<b>21</b>
2.1 The Continuum Equation	21
2.1.1 The Advective (Convective) Form	21
2.1.2 Dimensionless Forms and Limiting Cases of the Equation	23
2.1.3 The Divergence (Conservation) Form	27
2.1.4 Conservation Laws	28
2.1.5 Weak Forms of the PDE's/Natural Boundary Conditions	30
2.2 The Finite Element Equations/Discretization of the Weak Form	35
2.2.1 Advective Form	35
2.2.2 Divergence Form	42
2.2.3 Conservation Laws	42
2.2.4 An Absolutely Conserving Form	46
2.2.5 A Finite Difference Interpretation	50
2.2.6 A Control Volume FEM	52

---

<sup>†</sup> A \* indicates advanced or peripheral material.

2.3	Some Semi-Discrete Equations	56
2.3.1	One Dimension	56
a.	Linear elements	57
b.	Quadratic elements	63
2.3.2	Two Dimensions with Bilinear Elements	67
a.	An interior 4-patch	67
b.	A boundary 2-patch	74
c.	A boundary corner	77
d.	An internal line heat source	80
2.3.3	Two Dimensions with Biquadratic Elements	82
a.	An interior 4-patch	83
b.	An interior 2-patch	84
c.	An interior 1-patch	85
2.3.4	Two Dimensions with Serendipity Elements	86
a.	An interior 4-patch	86
b.	An interior 2-patch	87
c.	Another interior 2-patch	88
2.4	Open Boundary Conditions (OBC's)	91
2.4.1	One Dimension	91
2.4.2	Two Dimensions	102
2.5	Some Non-Galerkin Results	105
2.5.1	The Lumped Mass Approximation	105
2.5.2	One-point Quadrature	106
a.	An interior 4-patch of uniform rectangles	106
b.	A boundary 2-patch	108
c.	A boundary corner	108
2.5.3	Control Volume Finite Element (CVFEM)	109
a.	An interior 4-patch	109
b.	A boundary 2-patch	114
c.	A boundary corner	117
d.	OBC's	118
e.	A nine-node CVFEM	118
2.5.4	The Group FEM/Product Approximation	121
2.5.5	The Petrov–Galerkin FEM	122
2.6	Dispersion, Dissipation, Phase Speed, Group Velocity, Mesh Design, and—Wiggles	123
2.6.1	Qualitative Discussion	123
a.	Wiggles	123
b.	Dispersion	126
c.	Dissipation	127
d.	Phase speed	128
e.	Group velocity	129
f.	Mesh design	130
2.6.2	Quantitative Discussion for Some 1D Problems	131
a.	Pure advection with periodic BC's	132
b.	Advection-diffusion with periodic BC's	185
c.	Advection-diffusion with Dirichlet BC's	187
d.	Advection-diffusion with Dirichlet/ Neumann BC's	206
e.	Advection-diffusion with Neumann BC's at both ends	211
f.	Advection-diffusion with Dirichlet/ Robin BC's	212

g.	The advective-diffusive time scale	215
h.	Final remarks on 1D advection-diffusion	216
2.6.3	Extension to 2D	217
a.	Pure advection with periodic BC's	218
b.	Pure advection with Dirichlet BC's (inlet only)	222
c.	Advection-diffusion with Dirichlet BC's	226
d.	Advection-diffusion with periodic BC's	228
e.	Advection-diffusion with OBC's	229
f.	Final remarks on advection-diffusion via GFEM	229
2.7	Time Integration	230
2.7.1	Some Explicit ODE Methods	238
a.	Second-order Adams–Bashforth (AB2), an ‘explicit multi-step method’	238
b.	Third-order Adams–Bashforth (AB3), another ‘explicit multi-step method’	238
c.	Runge-Kutta methods (RK2, 4)	239
d.	Leapfrog (LF2; another explicit midpoint rule)	240
e.	Rational Runge-Kutta (RRK)	241
2.7.2	Application to Advection-Diffusion (Scalar Transport)	242
a.	Generalities	242
b.	Lumping the mass	243
c.	Stability estimates and the case for implicit methods	244
d.	Matrix method of stability analysis	249
e.	Balancing tensor diffusivity (BTD)	250
2.7.3	Some Implicit ODE Methods	255
a.	The trapezoid rule (TR)	257
b.	Implicit midpoint rule (IMR)	260
c.	Backward differentiation formulae (BDF)	263
2.7.4	A Variable-Step Implicit Method for Advection-Diffusion	264
a.	Variable step trapezoid rule	264
b.	Variable step backward Euler	268
c.	A model problem	268
d.	An aerospace version of TR	271
e.	TR on advection-diffusion	272
f.	The smoothing property	280
2.7.5	A Semi-Implicit Method	283
2.7.6	Dispersion ( <i>et al.</i> ) Errors For Some Fully Discrete Methods	286
a.	Introduction	286
b.	Semi-discrete the other way	287
c.	Fully discrete	289
d.	Generalizations and extensions	313
2.7.7	Other (Different) Methods Used by Others	315
a.	Methods based on trajectories/characteristics	316
b.	Methods based on modified equations	329
c.	Some least-squares finite element methods (LSFEM)	331

d.	Methods based on a discontinuous-in-time Galerkin ODE technique	333
e.	Methods based on least-squares and time-discontinuous ODE's	338
f.	A wave equation method	340
g.	Another combined method: Taylor least squares	340
2.7.8	Concluding Remarks and Suggestions	341
2.8	Additional Numerical Examples	342
2.8.1	Unstable ODE Example	342
2.8.2	Advection-Diffusion of a Puff (Point Source)	352
2.8.3	The Rotating Cone—A Pure Advection Test Problem	353
<b>Appendix 1 Some Element Matrices</b>		<b>357</b>
A.1.1	Advection–Diffusion Matrices	357
A.1.2	One-Dimensional Element Matrices	357
A.1.3	Two-Dimensional Element Matrices	358
A.1.4	Two-Dimensional Control Volume Finite Element Matrices	362
<b>Appendix 2 Further Comparison of Finite Elements and Finite Volumes</b>		<b>365</b>
A.2.1	Introduction	365
A.2.2	Viewpoint One	365
A.2.3	Viewpoint Two	373
<b>Appendix 3 Scalar Projections, Orthogonal and Not—and Projection Methods</b>		<b>379</b>
A.3.1	Introduction	379
A.3.2	Scalar Projections	383
A.3.2.1	The $L^2$ -Projection, $\mathcal{P}_0$	384
A.3.2.2	The $L^2$ -projection, $\mathcal{P}_0^h$	386
A.3.2.3	The $H^1$ -Projection, $\mathcal{P}_1$	393
A.3.2.4	The $H^1$ -projection, $\mathcal{P}_1^h$	402
A.3.2.5	The Projection Method	407
A.3.2.6	Brief Discussion of GFEM Errors on Elliptic BVP's	411
A.3.2.7	Numerical Examples	413
<b>References</b>		<b>423</b>
<b>Author Index</b>		<b>Ai-1</b>
<b>Subject Index</b>		<b>Si-1</b>