Contents

Preface	v
Acknowledgments	ix
L. Group and Hamiltonian Structures of Fluid Dynamics	1
§1. Symmetry groups for a rigid body and an ideal fluid	1
§2. Lie groups, Lie algebras, and adjoint representation	3
§3. Coadjoint representation of a Lie group	10
3.A. Definition of the coadjoint representation	10
3.B. Dual of the space of plane divergence-free vector fields	11
3.C. The Lie algebra of divergence-free vector fields and its	
dual in arbitrary dimension	13
§4. Left-invariant metrics and a rigid body for an arbitrary group	14
§5. Applications to hydrodynamics	19
§6. Hamiltonian structure for the Euler equations	25
§7. Ideal hydrodynamics on Riemannian manifolds	31
7.A. The Euler hydrodynamic equation on manifolds	31
7.B. Dual space to the Lie algebra of divergence-free fields	32
7.C. Inertia operator of an <i>n</i> -dimensional fluid	36
§8. Proofs of theorems about the Lie algebra of divergence-free	
fields and its dual	39
§9. Conservation laws in higher-dimensional hydrodynamics	42
§10. The group setting of ideal magnetohydrodynamics	49
10.A. Equations of magnetohydrodynamics and the	
Kirchhoff equations	49
10.B. Magnetic extension of any Lie group	50
10.C. Hamiltonian formulation of the Kirchhoff and	
magnetohydrodynamics equations	53
§11. Finite-dimensional approximations of the Euler equation	56
11.A. Approximations by vortex systems in the plane	56
11.B. Nonintegrability of four or more point vortices	58
11.C. Hamiltonian vortex approximations in three	
dimensions	59
11.D. Finite-dimensional approximations of diffeomorphism	•
groups	60

§12 .	The Navier-Stokes equation from the group viewpoint	63
П. Тор	ology of Steady Fluid Flows	69
§ 1.	Classification of three-dimensional steady flows	69
v	1.A. Stationary Euler solutions and Bernoulli functions	69
	1.B. Structural theorems	73
§2.	Variational principles for steady solutions and applications to	
	two-dimensional flows	75
	2.A. Minimization of the energy	75
	2.B. The Dirichlet problem and steady flows	78
	2.C. Relation of two variational principles	80
	2.D. Semigroup variational principle for two-dimensional steady flows	81
§3.	Stability of stationary points on Lie algebras	84
§4,	Stability of planar fluid flows	88
	4.A. Stability criteria for steady flows	89
	4.B. Wandering solutions of the Euler equation	96
§5.	Linear and exponential stretching of particles and rapidly	
	oscillating perturbations	99
	5.A. The linearized and shortened Euler equations	99
	5.B. The action-angle variables	100
	5.C. Spectrum of the shortened equation	101
	5.D. The Squire theorem for shear flows	102
	5.E. Steady flows with exponential stretching of particles	103
	5.F. Analysis of the linearized Euler equation	105
	5.G. Inconclusiveness of the stability test for space steady	400
87	flows	106
90.	Features of higher-dimensional steady flows	109
	6.A. Generalized Beltrami flows	109
	6.B. Structure of four-dimensional steady flows	111
	6.C. Topology of the vorticity function6.D. Nonexistence of smooth steady flows and sharpness of	112
	6.D. Nonexistence of smooth steady flows and sharpness of the restrictions	110
	the restrictions	116
III. Top	ological Properties of Magnetic and Vorticity Fields	119
§1.	Minimal energy and helicity of a frozen-in field	119
	1.A. Variational problem for magnetic energy	119
	1.B. Extremal fields and their topology	120
	1.C. Helicity bounds the energy	121
	1.D. Helicity of fields on manifolds	124
§2 .	Topological obstructions to energy relaxation	129
	2.A. Model example: Two linked flux tubes	129
	2.B. Energy lower bound for nontrivial linking	131
§3.	Sakharov–Zeldovich minimization problem	134

	Contents	xiii
§4.	Asymptotic linking number	139
•	4.A. Asymptotic linking number of a pair of trajectories	140
	4.B. Digression on the Gauss formula	143
	4.C. Another definition of the asymptotic linking number	144
	4.D. Linking forms on manifolds	147
§5.	Asymptotic crossing number	152
	5.A. Energy minoration for generic vector fields	152
	5.B. Asymptotic crossing number of knots and links	155
	5.C. Conformal modulus of a torus	159
§6.	Energy of a knot	160
	6.A. Energy of a charged loop	160
	6.B. Generalizations of the knot energy	163
§7.	Generalized helicities and linking numbers	1 66
	7.A. Relative helicity	166
	7.B. Ergodic meaning of higher-dimensional helicity	
	integrals	168
	7.C. Higher-order linking integrals	174
	7.D. Calugareanu invariant and self-linking number	177
	7.E. Holomorphic linking number	179
§8.	Asymptotic holonomy and applications	184
	8.A. Jones-Witten invariants for vector fields	184
	8.B. Interpretation of Godbillon-Vey-type characteristic	
	classes	191
IV. Diff	erential Geometry of Diffeomorphism Groups	195
§ 1.	The Lobachevsky plane and preliminaries in differential	
	geometry	196
	1.A. The Lobachevsky plane of affine transformations	196
	1.B. Curvature and parallel translation	197
	1.C. Behavior of geodesics on curved manifolds	201
	1.D. Relation of the covariant and Lie derivatives	202
§2.	Sectional curvatures of Lie groups equipped with a one-sided	
	invariant metric	204
§3.	Riemannian geometry of the group of area-preserving	
	diffeomorphisms of the two-torus	209
	3.A. The curvature tensor for the group of torus	
	diffeomorphisms	209
	3.B. Curvature calculations	212
§4.	Diffeomorphism groups and unreliable forecasts	214
	4.A. Curvatures of various diffeomorphism groups	214
	4.B. Unreliability of long-term weather predictions	218
§5.	Exterior geometry of the group of volume-preserving	.
	diffeomorphisms	219
§6.	Conjugate points in diffeomorphism groups	223

	§7.	Ġetti	ng around the finiteness of the diameter of the group of	
		volur	ne-preserving diffeomorphisms	225
		7.A.	Interplay between the internal and external geometry	
			of the diffeomorphism group	226
		7.B.	Diameter of the diffeomorphism groups	227
		7.C.	Comparison of the metrics and completion of the	
			group of diffeomorphisms	228
		7.D.	The absence of the shortest path	230
		7.E.	Discrete flows	234
		7.F.	Outline of the proofs	235
		7.G.	Generalized flows	236
		7.H.	Approximation of fluid flows by generalized ones	238
		7.I.	Existence of cut and conjugate points on	
			diffeomorphism groups	240
	§8.	Infini	ite diameter of the group of Hamiltonian diffeomorphisms	
		and s	ymplecto-hydrodynamics	242
		8.A.	Right-invariant metrics on symplectomorphisms	243
		8.B.	Calabi invariant	246
		8.C.	Bi-invariant metrics and pseudometrics on the group	
			of Hamiltonian diffeomorphisms	252
		8.D.	Bi-invariant indefinite metric and action functional on	
			the group of volume-preserving diffeomorphisms of a	
			three-fold	255
V.	Kine	matic	: Fast Dynamo Problems	259
	§1,	Dyna	amo and particle stretching	259
	•	1.A.	Fast and slow kinematic dynamos	259
		1.B.	Nondissipative dynamos on arbitrary manifolds	262
	§2.	Discr	ete dynamos in two dimensions	264
		2.A.	Dynamo from the cat map on a torus	264
		2.B.	Horseshoes and multiple foldings in dynamo	
			constructions	267
		2.C.	Dissipative dynamos on surfaces	271
		2.D.		273
	§3.	Main	antidynamo theorems	273
		3.A.	Cowling's and Zeldovich's theorems	273
		3.B.	Antidynamo theorems for tensor densities	274
		3.C.	Digression on the Fokker-Planck equation	277
		3.D.	Proofs of the antidynamo theorems	281
		3.E.	Discrete versions of antidynamo theorems	284
	§4.	Three	e-dimensional dynamo models	285
	_	4.A.	"Rope dynamo" mechanism	285
		4.B.	Numerical evidence of the dynamo effect	286

		Contents	χv
	4.C.	A dissipative dynamo model on a three-dimensional	
		Riemannian manifold	288
	4.D.	Geodesic flows and differential operations on surfaces	
		of constant negative curvature	293
	4.E.	Energy balance and singularities of the Euler equation	298
§5.	Dyna	mo exponents in terms of topological entropy	299
	5.A.		299
	5.B.	Bounds for the exponents in nondissipative dynamo	
		models	300
	5.C.	Upper bounds for dissipative L^1 -dynamos	301
VI. Dyn:	amica	l Systems with Hydrodynamical Background	303
§1.	The F	Korteweg-de Vries equation as an Euler equation	303
_	1.A.	. · ·	303
	1.B.	The translation argument principle and integrability of	
		the high-dimensional rigid body	307
	1.C.	Integrability of the KdV equation	312
	1.D.	Digression on Lie algebra cohomology and the	
		Gelfand-Fuchs cocycle	315
§2.	Equa	tions of gas dynamics and compressible fluids	318
	2.A.	Barotropic fluids and gas dynamics	318
	2.B.	Other conservative fluid systems	322
	2.C.	Infinite conductivity equation	324
§3.	Kähle	er geometry and dynamical systems on the space of	
	knots		326
		Geometric structures on the set of embedded curves	326
	3.B.	Filament, nonlinear Schrödinger, and Heisenberg	
		chain equations	331
	3.C.	Loop groups and the general Landau-Lifschitz	
		equation	333
		lev's equation	335
§ 5.		tic coordinates from the hydrodynamical viewpoint	340
	5.A.	Charges on quadrics in three dimensions	340
	5.B.	Charges on higher-dimensional quadrics	342
Reference	es		345
Index			369