CONTENTS

LIST OF TABLES xvi

LIST OF FIGURES xvi

INTRODUCTION: THE SUBJECT OF NEUTRON TRANSPORT THEORY xvii

PART I

GENERAL ANALYSIS

I. THE PHYSICAL PROPERTIES OF NEUTRONS 1

1.1. The properties underlying the concept of diffusion 1

1.2. The results of individual collisions 2

1.2.1. The relevant energy range. Thermal and non-thermal neutrons 2

1.2.2. Elastic scattering 3

1.2.3. Inelastic scattering 5

1.2.4. Fission 6

1.2.5. Prompt and delayed neutrons 8

1.3. The mean free path, the cross-section, the mean number of secondaries, etc. 8

1.3.1. Definitions and notations 8

1.3.2. General features of the energy dependence of cross-sections 10

1.4. Preview of the main problems of neutron transport theory 13

II. THE MATHEMATICAL FORMULATION OF THE LAWS OF NEUTRON MIGRATION 15

2.1. The form of the transport equation 15

2.2. Discussion of the form of the function $c_{t'}f(v'\Omega' \rightarrow v\Omega; t')$ 17

2.3. The boundary conditions 20

2.3.1. The interface between two media 21

2.3.2. The free surface of a medium 21

2.3.3. The condition at infinity 21

2.3.4. The initial conditions 22

2.4. The integral equation 22

2.5. The equivalence of the Boltzmann and integral equations 24

2.6. The case of re-entrant bodies and inhomogeneous media 26

2.7. The integral equation in the general case 27

III. STATIONARY AND TIME-DEPENDENT PROBLEMS. THE ADJOINT EQUATION 28

3.1. Stationary problems and critical-size problems 28

3.2. Time-dependent problems and their reduction to stationary problems 29

3.3. The adjoint equation and an orthogonality relation 31

3.4. The eigenfunction expansion of the solution of the inhomogeneous equation 33
3.5. Another orthogonality relation 34
3.6. The adjoint integral equation 36
3.7. Some further remarks 37

PART II

THE CONSTANT CROSS-SECTION APPROXIMATION

IV. ONE-GROUP THEORY AND ITS RANGE OF APPLICATION 39
4.1. The assumptions underlying one-group theory 39
4.2. Terminology 41
4.3. Other applications of the constant cross-section approximation 42
4.4. A note regarding thermal neutrons 43
4.5. The integral equation in the constant cross-section approximation 45
4.6. The adjoint integro-differential equation and the optical reciprocity theorem 47
4.7. The adjoint integral equation 49

V. EXACT SOLUTIONS FOR AN INFINITE MEDIUM WITH ISOTROPIC SCATTERING 51
5.1. The case of an infinite source-free medium 51
5.1.1. First method 51
5.1.2. Second method 52
5.1.3. Third method 53
5.2. The diffusion length 54
5.3. An isotropic point source in an infinite homogeneous medium 55
5.4. An anisotropic point source in an infinite homogeneous medium 59
5.5. An isotropic line or plane source in an infinite homogeneous medium 62
5.5.1. The line source 62
5.5.2. The plane source 64

VI. EXACT SOLUTIONS FOR AN INFINITE HALF-SPACE 67
6.1. Milne's problem 67
6.2. The asymptotic behaviour of the neutron flux in Milne's problem 72
6.3. The neutron current in Milne's problem 74
6.3.1. The neutron current in the deep interior 74
6.3.2. The neutron current at the free surface 76
6.4. Milne's problem in the presence of sources 77
6.5. The case of constant production 79
6.6. The albedo problem 81
6.7. The isotropic plane source case 83
6.8. The isotropic point source case 85

VII. THE CASE OF TWO ADJACENT HALF-SPACES 87
7.1. The extension of the Wiener–Hopf method 87
7.2. The asymptotic behaviour of the neutron flux 88
7.3. The angular distribution, flux, and current at the interface 90
7.4. The effects of sources 91

VIII. THE DIFFUSION APPROXIMATION 94
8.1. The basic ideas of diffusion theory 94
8.2. The problem of spherical systems 96
8.3. The boundary conditions at a free surface 97
8.4. The boundary conditions at an interface 98
8.5. Black bodies 100
8.6. Cavities and gaps 102
8.6.1. Cases where the presence of cavities has no effect 102
8.6.2. The case of a plane slit 103
8.6.3. Gaps and cavities of other shapes 105
8.7. The effects of sources 106
8.8. A remark on orthogonality 107

IX. THE SERBER–WILSON METHOD 109
9.1. Serber's formulation of the method 109
9.2. Wilson's formulation of the method 110
9.3. The explicit form of the Serber–Wilson conditions 112
9.4. The accuracy and limits of applicability of the method 113

X. THE SPHERICAL HARMONICS METHOD FOR PLANE GEOMETRIES 116
10.1. A general outline of the spherical harmonics method 116
10.2. The coefficients and exponents involved in the method 118
10.2.1. Auxiliary functions 118
10.2.2. Properties of the roots of the determinantal equation 122
10.2.3. Behaviour of the roots of the determinantal equation in high-order approximations 124
10.3. The boundary conditions 126
10.3.1. Conditions at an interface between media 126
10.3.2. Odd-order and even-order approximations 127
10.3.3. Conditions at infinity 128
10.3.4. Conditions at a free surface. Mark's lemma and boundary conditions 129
10.3.5. The application of Mark's boundary conditions 130
10.3.6. Conditions at a free surface (continued). Marshak's boundary conditions 133
10.3.7. Conditions at a surface exposed to neutrons 134
10.3.8. A thin absorbing slab at an interface between media 135
10.4. Multi-layer problems 137
10.5. The P_1 approximation 139
10.6. A remark on orthogonality 142

XI. THE SPHERICAL HARMONICS METHOD FOR SPHERICAL GEOMETRIES 146
11.1. The differential equations and their solution 146
11.2. Discussion of the solution 148
11.2.1. The reappearance of the functions $G_n(\nu)$ 148
11.2.2. The asymptotic angular distribution 149
11.3. The boundary conditions 151
11.3.1. Conditions at the origin 151
11.3.2. Boundaries already encountered in the plane case 152
11.3.3. Conditions at the surfaces of a gap 153

XII. THE SPHERICAL HARMONICS METHOD FOR OTHER GEOMETRIES 157
12.1. The general formulation of the spherical harmonics method 157
12.1.1. Preliminary 157
12.1.2. Notations and properties of spherical harmonics 158
12.1.3. The differential equations 159
12.1.4. The method of successive elimination 160
12.2. The reduction of the differential equations 163
12.2.1. The determination of the neutron flux 163
12.2.2. The neutron current and higher moments 164
12.2.3. The complementary functions 165
12.2.4. The boundary conditions 166
12.3. Cylindrical geometries 169
12.3.1. The choice of representation 169
12.3.2. The expressions for the moments 170
12.4. Yvon's modification of the spherical harmonics method 171

XIII. THE METHOD OF DISCRETE ORDINATES 174
13.1. General outline of the method 174
13.2. The roots of the characteristic equation 175
13.3. The boundary conditions. Elimination 177
13.4. The evaluation of the spherical harmonic moments 178
13.5. The solution for general μ. The iterated angular distribution 179
13.6. The choice of the summation formula 180
13.7. Comparison of the spherical harmonics and discrete ordinates methods 181
13.7.1. Plane geometries 181
13.7.2. Other geometries 181

XIV. THE PERTURBATION METHOD 183
14.1. First-order perturbations in critical-size problems 183
14.2. The variation 184
14.3. The transformation of the integrals 185
14.4. The statistical weight theorem and its form in the diffusion approximation 187
14.5. Applications of the statistical weight theorem 189

14.6. The perturbation in the neutron flux 190
14.7. The multiplication 192
14.8. Higher-order perturbations 193

XV. THE VARIATIONAL METHOD 195
15.1. Critical-size problems: the general theory 195
15.2. Examples of applications 198
15.3. Simplifications for slowly varying trial functions 200
15.4. Problems with inhomogeneous equations: the general theory 202
15.4.1. Finite systems 202
15.4.2. Infinite systems. The edge of the continuous spectrum 204
15.4.3. An alternative functional 207
15.5. Examples of applications 209
15.5.1. LeCaine's variational solution of Milne's problem 209
15.5.2. Marshak's determination of the linear extrapolation length of a black sphere 210
15.5.3. The neutron flux near a free surface 213
15.6. Miscellaneous remarks 215

XVI. THE ITERATION AND MONTE CARLO METHODS 218
16.1. The iteration method: a general survey 218
16.2. The combination of the iteration method with the discrete ordinates method or the spherical harmonics method 219
16.3. Iteration depending on a parameter. Miscellaneous remarks 220
16.4. The Monte Carlo method: a general survey 221
16.5. Comparison of the Monte Carlo and iteration methods 224
16.6. Statistical fluctuations and the choice of c' 225
16.7. Reduction of numerical work by using the symmetry of the system, etc. 225
16.8. Reduction of numerical work by the use of analytical methods 226
16.9. Reduction of sample sizes. Importance sampling 228
16.10. An alternative procedure 229
16.11. Conclusion 231

XVII. ANISOTROPIC SCATTERING 232
17.1. Preliminary results 232
17.2. The integral equation for the neutron flux 235
17.3. An infinite source-free medium 239
17.3.1. The formal solution 239
17.3.2. The case where c is close to unity. The transport mean free path and cross-section 240
17.3.3. The case where c is not close to unity 241
17.3.4. The number of roots of the equation for the diffusion length 242
17.4. Other exact solutions 243
17.4.1. An infinite medium containing sources 243
17.4.2. An infinite source-free half-space 244

17.5. Approximate methods 246
17.5.1. The diffusion approximation and the Serber–Wilson method 246
17.5.2. The spherical harmonics method 247
17.5.3. The discrete ordinates method 248

PART III

ENERGY-DEPENDENT PROBLEMS WITH SPECTRUM REGENERATION

XVIII. A GENERAL SURVEY OF ENERGY-DEPENDENT PROBLEMS 251
18.1. Slowing-down and spectrum-regenerating media 251
18.2. Media which do not degrade the spectrum 252
18.3. Spectrum-regeneration problems and slowing-down problems 253
18.4. The main methods of solution of spectrum-regeneration problems 254

XIX. MULTI-GROUP THEORY 255
19.1. A general survey 255
19.2. The assumptions and basic equations of multi-group theory 255
19.2.1. The formal derivation of the equations 255
19.2.2. The validity of the multi-group approximation for an infinite source-free medium 257
19.2.3. The validity of the multi-group approximation for finite systems. General conclusions 258
19.2.4. The matrix form of the multi-group equations 261
19.2.5. The adjoint equations 262
19.3. Infinite and semi-infinite media 263
19.3.1. An infinite source-free medium 263
19.3.2. 'Solutions' which do not satisfy the conditions at infinity 267
19.3.3. An infinite medium containing sources 269
19.3.4. Semi-infinite media 270
19.4. The diffusion approximation and the Serber–Wilson method 272
19.4.1. General remarks 272
19.4.2. The case where the infinite-medium equation has enough solutions 273
19.4.3. The case where the infinite-medium equation has not enough solutions 274
19.5. The spherical harmonics method and the discrete ordinates method 277
19.5.1. The spherical harmonics method for plane geometries 277
19.5.2. The spherical harmonics method for other geometries 279
19.5.3. The P_1 approximation and a comparison of it with the diffusion approximation 280
19.5.4. The discrete ordinates method 281
19.6. The perturbation method 282
19.7. The Monte Carlo method and anisotropic scattering 283

XX. THE METHOD OF POLYNOMIAL APPROXIMATIONS 285
20.1. The type of problem for which the method is intended 285
20.2. The transformation of the Boltzmann equation 286
20.2.1. Simplifications made in the presentation of the method 286
20.2.2. The application of the spherical harmonics method 287
20.3. The method of polynomial approximations 288
20.3.1. The basic idea of the method 288
20.3.2. The application of Mellin transforms 289
20.3.3. The determinantal equation 292
20.4. Extensions and modifications of the method 293
20.4.1. The behaviour of the exact solution for small energies 293
20.4.2. The extension of the method to cases where low-energy neutrons predominate 295
20.4.3. The modification of the method to take account of thermal neutrons 297

XXI. FEYNMAN'S METHOD 302
21.1. A single homogeneous body with isotropic scattering 302
21.1.1. General outline of the method 302
21.1.2. The coefficients $A_{j'\to j}(v' \to v)$ in (21.6) 304
21.1.3. The energy-dependent factors $p_j(v)$ 305
21.1.4. The critical-size problem 307
21.1.5. A remark on the solution of the auxiliary one-group problem 307
21.2. A reactive core in an elastically scattering reflector 308
21.2.1. Formulation of the problem 308
21.2.2. The application of the eigenfunction expansion 309
21.3. An inelastically scattering reflector 310
21.4. Feynman's method for anisotropic scattering 314
21.4.1. The case of a bare slab. Transformation of the equations 314
21.4.2. The application of the eigenfunction expansion 316
21.4.3. The solution of the auxiliary one-group problem 317
21.4.4. Bodies of other shapes and those surrounded by a reflector which scatters only elastically 318

PART IV

SLOWING-DOWN PROBLEMS

XXII. A GENERAL SURVEY OF SLOWING-DOWN PROBLEMS. THE SPATIAL MOMENTS OF THE NEUTRON DISTRIBUTION AS FUNCTIONS OF ENERGY 321
22.1. General survey 321
22.1.1. The main types of slowing-down problem 321
22.1.2. The form of the Boltzmann equation in slowing-down problems 323
22.1.3. The lethargy and the collision interval 326
22.1.4. The slowing-down density 327

22.2. The integral of the neutron energy spectrum over all space 329
22.2.1. A single element with no capture 329
22.2.2. A mixture with no capture 333
22.2.3. The effects of capture 334
22.2.4. The case of $1/v$ capture 338
22.3. The spatial moments of the neutron distribution. The slowing-down length 340
22.3.1. Definitions 340
22.3.2. The equations for the spatial moments 342
22.3.3. The second spatial moment in a single element without capture 344
22.3.4. The case of a mixture and the higher spatial moments 345

XXIII. AGE THEORY 346
23.1. The assumptions underlying age theory 346
23.2. The basic equation of age theory 348
23.2.1. The reduction of the Boltzmann equation 348
23.2.2. The equation for the slowing-down density 350
23.2.3. The age of the neutrons 352
23.2.4. Modified age theory. The effects of resonance capture 353
23.3. The range of applicability of age theory 355
23.4. The boundary conditions and methods of solution of the age equations 357
23.4.1. The boundary conditions 357
23.4.2. Analytical methods of solution of the age equations 358
23.4.3. Numerical methods; the direct approach 359
23.5. Application of age theory to spectrum-regeneration problems 360
23.5.1. Modified two-group theory: general considerations 360
23.5.2. Modified two-group theory: the form of the equations 361
23.5.3. The infinite-medium solution 364
23.5.4. A comparison with ordinary multi-group theory 365
23.6. Improvements in age theory. Other methods for moderate distances 368

XXIV. SLOWED-DOWN NEUTRONS AT LARGE DISTANCES FROM THE SOURCE. THE CASE OF CONSTANT CROSS-SECTIONS 370
24.1. A qualitative discussion 370
24.2. The formal solution 372
24.2.1. The method of Fourier–Mellin transforms 372
24.2.2. The inversion of the Fourier transform 373
24.2.3. The inversion of the Mellin transform. The method of steepest descent 374
24.2.4. Discussion. An alternative form of the results 376
24.3. Preliminary survey of the more effective analytical methods available 377
24.4. Bethe's method 378
24.4.1. General outline of the method 378
24.4.2. The evaluation of the residue 380
24.4.3. A single fairly heavy element 382
24.4.4. The connexion with age theory 383

24.5. Wick's asymptotic solution for constant cross-sections 384
24.5.1. The approximate form of the equation 384
24.5.2. The application of Bessel transforms 387
24.5.3. The solution of the transformed equation: the homogeneous case 388
24.5.4. The evaluation of the residue. Final results 390

XXV. SLOWED-DOWN NEUTRONS AT LARGE DISTANCES FROM THE SOURCE. THE CASE OF VARIABLE CROSS-SECTIONS 392
25.1. Preliminary survey 392
25.2. Singularities of the Fourier–Mellin transform of the density when the cross-section varies monotonically 394
25.3. The method of Verde and Wick 396
25.3.1. The singularities 396
25.3.2. The evaluation of the coefficients 398
25.3.3. The inversion of the Fourier transforms 399
25.4. Wick's asymptotic solution 401
25.4.1. The reduction of the equations 401
25.4.2. An eigenvalue problem 403
25.4.3. Final results 404
25.5. Hydrogenous media: the method of Spencer and Fano 407
25.5.1. Introduction 407
25.5.2. Spatial moments and orthogonal polynomials in spatial coordinates 408
25.5.3. The choice of the weight function 410
25.5.4. The polynomials $U_s(y)$ and $V_s(y)$ 412
25.5.5. The equations for the moments 415

XXVI. HOLTE'S METHOD 417
26.1. Holte's method 417
26.2. The case of constant mean free path 417
26.2.1. The determination of the transforms 417
26.2.2. The neutron density 422
26.3. The case of variable mean free path 423
26.3.1. A survey of the singularities 423
26.3.2. The determination of the transform of the solution, for small p 425
26.3.3. The case of moderate distances from the source when the mean free path varies only slightly 426
26.3.4. The application of the method of steepest descent 428
26.4. The case where the mean free path decreases with the velocity 429
26.4.1. Preliminary transformations 429
26.4.2. The behaviour of the integrand 430
26.4.3. The evaluation of the integral 431

APPENDIX A: Some remarks regarding the completeness of the sets of eigenfunctions $N_{t,n}(\mathbf{r},v\Omega)$ and $N_{c,n}(\mathbf{r},v\Omega)$ 435

APPENDIX B: Carlson's method 437

BIBLIOGRAPHY 439

INDEX OF SYMBOLS 442

INDEX OF NAMES 444

INDEX OF SUBJECTS 445

LIST OF TABLES

I. Comparison of critical radii as given by various approximate methods 114

II. Values of ν_i for $N = 1, 3, 5, 7$ 119

III. Values of $G_n(\nu_i)$ for $N = 1, 3, 5, 7$ 120

IV. Values of the functions used in Bethe's method 383

LIST OF FIGURES

1. Energy spectrum of fission neutrons 7

2*a*. Variation of the total cross-section of indium with energy 12

2*b*. Variation of the total cross-section of hydrogen (in paraffin) with energy 12

3. Path of integration used in deriving (5.25) 57

4. Neutron density for an isotropic point source in an infinite homogeneous medium with $c = 1$ (A), $c = 0{\cdot}81$ (B), $c = 0{\cdot}5$ (C) 58

5*a*, 5*b*. Paths of integration used in deriving (6.9) 70

6. Variation of extrapolation length with radius for a black sphere (A) and for a black cylinder (B) 102