Contents

Chapt	er	Page
	Preface	v
	Part I. Qualitative Methods	
1	PHASE PLANE; SINGULAR POINTS	3
	1. Introductory remarks	3
	2. Theorem of Cauchy-Lipschitz; generalities	4
	3. Phase plane	8
	4. Singular points; elementary singular points	10
	5. Examples of singular points of a linear d.e.	14
	6. Canonical transformation; abridged equations	19
	7. Distribution of singular points; parameter space	26
	8. Center	32
	9. Certain conclusions	38
2	NONLINEAR CONSERVATIVE SYSTEMS	40
	1. Introductory remarks	40
	2. Fundamental properties of nonlinear conservative systems	42
	3. Motions in the large; separatrix	45
	4. Effect of a parameter in a differential equation; bifurcation	
	values	47
	5. Problem of the rotating pendulum	48
	6. Attraction of current-carrying conductors	51
	7. Further properties of conservative systems; Hamiltonian	
	variables; integral invariants	53
	8. Oscillating circuit with no resistance but with a nonlinear	
	inductance	62
	9. Volterra's problem	65

Chapt	er	Page
3	LIMIT CYCLES OF POINCARÉ	71
	1. Definitions	71
	2. Examples of limit cycles	72
	3. Physical significance of limit cycles	74
	4. Polycyclic configurations	75
	5. The index of Poincaré	77
	6. Brouwer's fixed point theorem	80
	7. Negative criterion of Bendixson	82
	8. Poincaré-Bendixson theorem	84
	9. Cycles without contact; curve of contacts	85
	10. Complete topological configurations; singular points at infinity	91
	11. Nonanalytic cycles	96
	12. Topological configurations	98
4	GEOMETRICAL ANALYSIS OF PERIODIC SOLUTIONS	3 101
	1. Introductory remarks	101
	2. Liénard's theory; curves Γ and Δ ; criterion of periodicity	101
	3. Liénard's phase plane; graphical construction of curves Γ	108
	 Asymptotic cases of Liénard's equation 	111
	5. Principle of symmetry	113
	6. Energy criterion in a nearly linear case	116
5	STABILITY (VARIATIONAL EQUATIONS, CHAR-	
	ACTERISTIC EXPONENTS)	118
	1. Introductory remarks	118
	2. Definition of stability (Liapounov)	118
	3. Variational equations	121
	4. Variational equations of singular points	123
	5. Variational systems with constant coefficients	124
	6. Linear systems with periodic coefficients	127
	7. Orbital stability	130
6	STABILITY (SECOND METHOD OF LIAPOUNOV)	134
	1. Introductory remarks	134
	2. Theorems of Liapounov (stability)	136
	3. Geometrical interpretation of Liapounov's theorems	138
	4. Certain auxiliary propositions concerning the functions V	140

х

Chapter		Page
5.	Construction of the function V for a differential system	
	with constant coefficients	143
6.	Stability on the basis of abridged equations	144
7.	Certain generalizations	145
8.	Aiserman's problem	147
9.	Critical cases	150
10.	Functions V containing time explicitly	154
11.	Theorems of Liapounov for functions V containing t explicitly	155
12.	Criteria of stability for equations periodic in t	156
	Application to control theory	158
	Concluding remarks	161
	0	
7 TI	IEORY OF BIFURCATIONS	163
1.	Introductory remarks	163
	Successor functions; geometry of bifurcation effects	165
3.	Bifurcation of a cycle from a focus	169
	Applications of the bifurcation theory	173
	Other bifurcation problems	177
	Examples of coalescing limit cycles	178
	Algebraic approach to the bifurcation theory	178
8.	Bifurcation diagrams; effect of bifurcation points on structures	182
0	Noncritical systems	182
	Remarks	185
10.	Remarks	100
8 CY	LINDRICAL AND TOROIDAL PHASE SPACES	190
1	Introductory remarks	190
	Differential equation of an electromechanical system	193
	Cylindrical phase trajectories of a conservative system	195
	Cylindrical phase trajectories of nonconservative systems	197
	Topological configurations on the cylinder	202
	Oscillations of a synchronous motor	202
	Toroidal phase surface	204
	Examples of nonrecurrent trajectories	206
	Gliding flight	207

Part II. Quantitative Methods

INTRODUCTION

211

Chapter	Page
9 PERTURBATION METHOD	217
1. Secular terms	217
2. Energy fluctuations in a van der Pol oscillator	219
3. Lindstedt's method	224
4. Theorem of Poincaré	228
10 PERIODIC SOLUTIONS (POINCARÉ)	232
1. Introductory remarks	232
2. Nonresonance oscillation of an (NA) system	234
3. Resonance oscillation of an (NA) system	236
4. Calculation of periodic solutions; example	240
5. Autonomous systems	243
6. Calculation of periodic solutions of autonomous syst	ems 246
7. Nonanalytic case	250
11 OSCILLATIONS IN SYSTEMS WITH SEVER	RAL
DEGREES OF FREEDOM	252
1. Introductory remarks	252
2. Periodic solutions of homogeneous linear systems	-
constant coefficients	254
3. Nonresonance oscillations of nonautonomous systems	
4. Resonance oscillations of nonautonomous systems	260
5. Periodic resonance solutions of nonautonomous syst	tems
with nonanalytic d.e.	263
6. Oscillations of autonomous systems	267
7. Self-excited oscillations in coupled circuits	270
8. Method of averaging	273
12 ALMOST PERIODIC OSCILLATIONS IN NEAD	RLY
LINEAR SYSTEMS	282
1. Introductory remarks	282
2. Almost periodic solutions	285
3. Existence of A.P.S. in noncritical cases	289
4. Transformation of Krylov-Bogoliubov; standard syste	
5. Almost periodic solutions of standard systems	291
6. Almost periodic oscillations when all roots are critical	
7. General case: combination of critical and noncritical	
8. Van der Pol equation with two forcing terms	296

xii

Chapte	27	Page
_	9. Parameters of the generating system; nonresonance and	
	resonance frequencies	299
	10. Forced oscillations of a mono-rail car	301
	11. Physical aspects of A.P. oscillations	306
13	DETERMINATION OF CHARACTERISTIC EXPO-	
	NENTS	308
	1. Determination of characteristic exponents on the basis of	
	Poincaré's theory	308
	2. Stability of periodic solutions	313
	3. Determination of characteristic exponents by approxima-	
	tions	314
	4. Second-order systems; invariants A_i	317
	5. Zones of stability	319
	6. Calculation of zones of stability	322
14	ASYMPTOTIC METHODS OF KRYLOV-BOGOLIU-	
	BOV-MITROPOLSKY (AUTONOMOUS SYSTEMS)	329
	1. Introductory remarks	329
	2. Successive approximations for the autonomous systems	329
	3. Differential equations of the first approximation	333
	4. Nonlinear damping	338
	5. Self-excited systems	341
	6. Stationary amplitudes and their stability	343
	7. Equivalent linearization	348
15	ASYMPTOTIC METHODS OF KRYLOV-BOGOLIU-	
	BOV-MITROPOLSKY (NONAUTONOMOUS SYSTEMS)	356
	1. Introductory remarks	356
	2. Formulation of the problem	358
	3. Successive approximations in nonresonance cases	360
	4. Successive approximations for resonance oscillations	367
	5. External periodic excitation of a nonlinear oscillator;	
	jumps of amplitude	375
	6. Nonstationary processes; slow time	380
	7. Successive approximations for nonstationary processes;	
	slow time	383
	8. Oscillations of a pendulum with a variable length	387

xiii

Chapter	Page
16 STROBOSCOPIC METHOD	390
1. Introductory remarks	390
2. Transformation of points and regions; planes (ψ) and (φ);	
stroboscopic image	390
3. Stroboscopic differential equations	395
4. Application of the stroboscopic method to the Mathieu	
oscillators	401
5. Application of the method to autonomous systems; second	
approximation	406
6. Further properties of the stroboscopic transformation	409
7. Existence and stability of the fixed point	411
17 GENERALIZATION OF NYQUIST'S DIAGRAM FOR	
NONLINEAR SYSTEMS	416
1. Introductory remarks	416
· · · · · · · · · · · · · · · · · · ·	

1. Introductory remarks	410
2. Theory of generalized Nyquist diagrams (Theodorchik)	419
3. Stationary state of self-excitation	422
4. Interaction of nonlinear oscillations	425
5. Stability	426
6. Retarded actions	428
7. Concluding remarks	431

Part III. Oscillations of Nearly Linear Systems

INTRODUCTION	433
--------------	-----

18	SYNCHRONIZATION	438
	1. Introductory remarks	438
	2. Theory of van der Pol	439
	3. Topological analysis of Andronov and Witt	441
	4. Conditions of the stationary state of synchronization	443
	5. Theory of synchronization by the stroboscopic method	444
	6. Mutual synchronization	448
	7. Other forms of synchronization	455
19	NONLINEAR RESONANCE	460
	1. Introductory remarks	460
	2. Subharmonics	462

xiv

Chapt	er	Page
	3. Theory of L. Mandelstam and N. Papalexi	464
	4. Application to the resonance of the order $\frac{1}{2}$	469
	5. Subharmonic resonance by the stroboscopic method	473
	6. Applications of the stroboscopic method	476
	7. Zones of subharmonic resonance	482
	8. Comparison of different methods	484
20	PARAMETRIC EXCITATION	488
	1. Introductory remarks	488
	 General form of differential equation of parametric action Special cases—bifurcation surfaces; stability of the state 	489
	of rest	492
	4. Phenomenon of Bethenod	495
		498
	 Origin of the parametric action Parametric excitation in electrical circuits 	503
	7. Autoparametric excitation	505
	8. Parametric excitation by the asymptotic method	509
	8. Farametric excitation by the asymptotic method	509
21	OSCILLATIONS CAUSED BY RETARDED ACTIONS	514
	1. Introductory remarks	514
	2. Difference-differential equations arising in applications	516
	3. Characteristic equation; neighborhood of a harmonic solution	518
	4. Advanced versus retarded actions	525
	5. Nonlinear problem; stationary state; frequency correction	
	and stability	526
	6. On the physical nature of retarded actions	532
	7. Experimental evidence; electronic analogue	534
	8. Econometric and other problems	537
22	TOPOLOGY OF LIÉNARD'S EQUATION IN A PARA-	
	METER SPACE	541
	1. Introductory remarks	541
	2. Formation of the stroboscopic equation	543
	3. Phase portraits of Liénard's equation	545
	4. Bifurcations of the third kind; more general cases	550
	5. Special cases of Liénard's equation	551
	6. Phase portraits of Rayleigh's and mixed equations	555
	7. Frequency correction	556

xv

Chapt		Page
	8. Special forms of the van der Pol equation	557
	9. Certain physical considerations	559
23	INTERACTION OF NONLINEAR OSCILLATIONS	562
	1. Introductory remarks	562
	2. The van der Pol theory of interaction	563
	3. Interaction of two autoperiodic oscillations	566
	4. Analysis of stability of singular points	567
	5. Special cases.	570
	6. Remarks	571
24	ASYNCHRONOUS ACTIONS	572
	1. Introductory remarks	572
	2. Small external periodic excitation	574
	3. Asynchronous quenching	576
	4. Finite external periodic excitation	577
	5. Asynchronous excitation	580
	6. Concluding remarks	582
25	SYSTEMS WITH INERTIAL NONLINEARITIES	585
~~~		585
	1. Introductory remarks 2. Inertial nonlinearity	585
	3. Van der Pol oscillator with a conductor $R(x_0^2)$	588
	4. Oscillations produced by nonlinear conductors; physical	200
	considerations	591
	5. General problem	592
	6. Stability; concluding remarks	595
	o. Stability, concluding remarks	575
	Part IV. Relaxation Oscillations	

### INTRODUCTION 599 26 DISCONTINUOUS THEORY OF RELAXATION OSCILLATIONS 605 1 Piecewise analytic phenomena 605

1. Piecewise analytic phenomena	605
2. Degeneration theory and its physical significance	609
3. Conditions imposed by invariants	614
4. Discontinuous theory of Mandelstam-Chaikin	615

xvi

Chapter		Page
27	APPLICATION OF THE DISCONTINUOUS THEORY	
	TO ELECTRICAL PROBLEMS	618
	1. Degenerate RC oscillator	618
	2. Oscillator with two degrees of freedom with degeneration	
	in each degree	621
	3. Connection between critical points and piecewise analytic	
	phenomena	626
	4. Symmetrical multivibrator circuits	627
	5. Concluding remarks	639
28	APPLICATION OF THE DISCONTINUOUS THEORY	
	TO MECHANICAL PROBLEMS	632
	1. Introductory remarks	632
	2. Mechanical relaxation oscillations	633
	3. Relaxation oscillations of a Prony brake	633
	4. Analogy between mechanical and electrical relaxation	
	oscillations	637
	5. Clocks	639
	6. Froude's pendulum	644
29	DISCONTINUOUS THEORY OF VOGEL	648
	1. Introductory remarks	648
	2. Fundamentals of the theory	649
	3. Analytic formulation; a special case	655
	4. Physical interpretation	656
	5. A numerical example	657
	6. Multivibrator	658
	7. A further extension of the theory	661
	8. Concluding remarks	662
30	ASYMPTOTIC METHODS	664
	1. Introductory remarks	664
	2. Asymptotic theory versus discontinuous theory	665
	3. Asymptotic theory applied to an asymmetrical multi-	
	vibrator	670
	4. Method of Cartwright-Littlewood	672
	4. Method of Cartwright-Littlewood 5. Asymptotic expansions	672 676

Chapter	
31 PIECEWISE LINEAR IDEALIZATION	687
1. Introductory remarks	687
2. Point transformation method	688
3. Calculation of a piecewise linear limit cycle	690
4. Successor function and the fixed point	692
5. Successor function and the fixed point (hard self-	
excitation)	695
6. Topology of certain relay systems	699
7. Point transformation for $\beta < 1$	701
8. Nonanalytic cycles and their stability	703
9. Remarks	705
10. Concluding remarks	706

xviii