Table of Contents

Preface iii

Introduction xi

Chapter I. MULTIPLICITY OF ROOTS OF FUNCTIONS AND MULTIPLICITY OF INTERSECTION POINTS OF TWO CURVES . . . 1

Introduction 1

§1. Multiplicity of a root of a function 1

1. δ-closeness to rank r (1). 2. The theorem of a small increment of implicit functions (3). 3. Root multiplicity of a function of a single variable (7). 4. Multiplicity of a root relative to a given class of functions (13).

§2. The multiplicity of a common point of two curves 14

1. Definition of multiplicity (14). 2. Condition of simplicity for an intersection point of two curves (15). 3. Condition of duplicity for an intersection point of two curves (17).

Chapter II. DYNAMIC SYSTEMS CLOSE TO A GIVEN SYSTEM AND PROPERTIES OF THEIR PHASE PORTRAITS 23

Introduction . . . . .

§3. Closeness of solutions. Regular transformation of close systems 23

1. Theorems of closeness of solutions (23). 2. ϵ-closeness of regions. Lemmas of regular transformation (27).

§4. Intersection of paths of close systems with arcs and cycles without contact 30

1. Intersection with one arc without contact (30). 2. Paths of close systems between two arcs without contact (39).

Chapter III. THE SPACE OF DYNAMIC SYSTEMS AND STRUCTURALLY STABLE SYSTEMS 50

Introduction 50

§5. The space of dynamic systems 51

1. The space of dynamic systems in a plane region (51). 2. The space of dynamic systems on a sphere (52).

§6. Definition of a structurally stable dynamic system 55

1. Dynamic systems on a plane (55). 2. Structurally stable systems on a sphere (58). 3. Structural stability of dynamic systems in $R_k^{(r)}$ and $R_a^{(r)}$ (59).

§ 7. Structurally stable and structurally unstable paths. Necessary condition of structural stability of an equilibrium 62

1. Structurally stable and structurally unstable paths (62). 2. Finite number of equilibrium states in a structurally stable system (63). 3. Multiplicity of an equilibrium state (65).

Chapter IV. EQUILIBRIUM STATES OF STRUCTURALLY STABLE SYSTEMS. SADDLE-TO-SADDLE SEPARATRIX 68

Introduction 68

§ 8. Structural stability of a node and a simple focus 68

1. Canonical system (68). 2. Structural stability of a simple node and a focus (71).

§ 9. Structural stability of a saddle point 78

1. Reduction of the system to canonical form by a nearly identical transformation (78). 2. Proof of the structural stability of a saddle point (80).

§10. Structural instability of an equilibrium state with pure imaginary characteristic roots 89

1. Investigation of an equilibrium state with complex characteristic roots (a review) (89). 2. Calculation of the first focal value (92). 3. The theorem of the creation of a closed path from a multiple focus (93). 4. Proof of structural instability (95).

§11. A saddle-to-saddle separatrix..... 97

1. The behavior of the separatrix under vector field rotation (97). 2. Proof of structural instability (100).

Chapter V. CLOSED PATHS IN STRUCTURALLY STABLE SYSTEMS 103

Introduction 103

§12. A closed path and its neighborhood. Succession function ... 104

1. Introduction of the succession function (104). 2. The configuration of paths in the neighborhood of a closed path (105). 3. The case of an analytical dynamic system (108). 4. The case of a nonanalytical dynamic system (109).

§13. Curvilinear coordinates in the neighborhood of a closed path. Succession function on a normal to a path 110

1. Curvilinear coordinates in the neighborhood of a closed path (110). 2. Transformation to the variables s, n in a dynamic system (113). 3. Succession function on a normal to a closed path (116).

§14. Proof of structural stability of a simple limit cycle 118

§ 15. Structurally unstable closed paths 124

1. The fundamental lemma (124). 2. The theorem of the creation of a closed path from a multiple limit cycle (127). 3. Structural instability of a closed path with a zero characteristic index (133).

Chapter VI. NECESSARY AND SUFFICIENT CONDITIONS OF STRUCTURAL STABILITY OF SYSTEMS 136

Introduction 136

§ 16. Singular paths and semipaths of dynamic systems 137

1. Finite number of closed paths for structurally stable systems (137). 2. Regions with normal boundary (139).

§ 17. A regular system of neighborhoods and the partition of $\bar{G}^*$ into canonical neighborhoods and elementary quadrangles 142

1. A regular system of canonical neighborhoods for structurally stable systems (142). 2. The partition of the region $\bar{G}^*$ into canonical neighborhoods and elementary quadrangles (146).

§ 18. The fundamental theorem of structural stability of a dynamic system 150

1. Lemmas (150). 2. The fundamental theorem for a plane region (156). 3. The fundamental theorem for a sphere (162). 4. Remarks and supplements (166).

Chapter VII. CELLS OF STRUCTURALLY STABLE SYSTEMS. AN ADDITION TO THE THEORY OF STRUCTURALLY STABLE SYSTEMS. 174

Introduction 174

§ 19. Cells of structurally stable dynamic systems 175

1. General considerations pertaining to cells of dynamic systems (175). 2. Doubly connected cells of structurally stable systems (176). 3. Interior cells of structurally stable systems. Simply connected interior cells (179).

§ 20. Examples of structurally stable systems 190

§ 21. A definition of structural stability foregoing the requirement of ϵ-identity 195

Chapter VIII. BIFURCATIONS OF DYNAMIC SYSTEMS. DECOMPOSITION OF A MULTIPLE EQUILIBRIUM STATE INTO STRUCTURALLY STABLE EQUILIBRIUM STATES 202

Introduction 202

§ 22. The degree of structural instability and bifurcations of dynamic systems 203

§ 23. Decomposition of a multiple equilibrium state into structurally stable equilibrium states 218

1. The number of structurally stable equilibrium states obtained from a multiple equilibrium state (218). 2. The character of the structurally stable equilibrium states obtained from a multiple equilibrium state with $\sigma \neq 0$ (222). 3. The character of the structurally stable equilibrium states obtained from a multiple equilibrium state with $\sigma = 0$ (227).

Chapter IX. CREATION OF LIMIT CYCLES FROM A MULTIPLE FOCUS 238

Introduction 238

§ 24. Focal values 239

1. Some properties of the succession function (239). 2. Multiplicity of a multiple focus. Focal values (242). 3. Calculation of the focal values of a multiple focus (244). 4. The case of an analytical system (249).

§ 25. Creation of limit cycles from a multiple focus 254

1. The fundamental theorem (254). 2. Bifurcations of a dynamic system in the neighborhood of a multiple focus (259). 3. Bifurcations in the neighborhood of a multiple focus of multiplicity 1 (261).

Chapter X. CREATION OF CLOSED PATHS FROM A MULTIPLE LIMIT CYCLE 266

Introduction 266

§ 26. Expressions for the derivatives of the succession function. Multiplicity of a limit cycle 267

1. Expressions for the derivatives of succession functions (267). 2. Multiplicity of a limit cycle (272).

§ 27. Creation of limit cycles from a multiple limit cycle 277

1. The fundamental theorem (277). 2. Supplements (282).

Chapter XI. CREATION OF LIMIT CYCLES FROM THE LOOP OF A SADDLE-POINT SEPARATRIX 286

Introduction 286

§ 28. Auxiliary material 287

1. Correspondence function and succession function (287). 2. Some properties of a saddle point and its separatrices (295).

§ 29. Creation of limit cycles from the separatrix loop of a simple saddle point 299

1. Some properties of the separatrix loop (299). 2. Theorems of the creation of a closed path from a separatrix loop (307). 3. The uniqueness of the closed path created from a separatrix loop (311). 4. The case $P'_x(x_0, y_0) + Q'_y(x_0, y_0) = 0$ (317).

Chapter XII. CREATION OF A LIMIT CYCLE FROM THE LOOP OF A SADDLE-NODE SEPARATRIX. SYSTEMS OF FIRST DEGREE OF STRUCTURAL INSTABILITY AND THEIR BIFURCATIONS . . . 322

Introduction 322

§ 30. Creation of a limit cycle from the loop of a saddle-node separatrix 323

1. The existence theorem (323). 2. The uniqueness theorem (326).

§ 31. Dynamic systems of the first degree of structural instability and their bifurcations. 330

1. The definition of a system of the first degree of structural instability (330). 2. Equilibrium states of systems of the first degree of structural instability (331). 3. Closed paths of systems of the first degree of structural instability (340). 4. A saddle-point separatrix forming a loop (349). 5. The simplest structurally unstable paths (351). 6. The properties of the separatrices of a saddle-node in systems of the first degree of structural instability (364). 7. Properties of separatrices of saddle points of systems of the first degree of structural instability (367). 8. The fundamental theorem (the necessary and sufficient conditions for systems of the first degree of structural instability) (374). 9. Bifurcations of systems of the first degree of structural instability (375).

Chapter XIII. LIMIT CYCLES OF SOME DYNAMIC SYSTEMS DEPENDING ON A PARAMETER 377

Introduction 377

§ 32. The behavior of limit cycles of some dynamic systems following small changes in the parameter 378

1. The succession function in the neighborhood of a closed path (378). 2. Statement of the problem (385). 3. Newton's polygon and solution of the equation $F(w, z) = 0$ (387). 4. The behavior of limit cycles of some dynamic systems following small changes in the parameter (396).

§ 33. Creation of a limit cycle from a closed path of a conservative system 402

1. The integral invariant and conservative system. Statement of the problem. The method of the small parameter (402). 2. Systems close to a linear conservative system (409). 3. The general case of a system close to a conservative system (414). 4. Systems close to a Hamiltonian system (418).

Chapter XIV. THE APPLICATION OF THE THEORY OF BIFURCATIONS TO THE INVESTIGATION OF PARTICULAR DYNAMIC SYSTEMS 423

Introduction 423

§ 34. Examples 424

Appendix 462

1. Theorems of the continuous dependence of the solutions of a system of differential equations on the right-hand sides and of the differentiability of solutions (462). 2. A proposition regarding functions of many variables (468). 3. The lemma about the normals of a simple smooth closed curve (469). 4. Proof of the differentiability of the function $R(\rho, \theta)$ with respect to ρ (471). 5. A remark concerning the definition of a structurally stable dynamic system (477).

Bibliography. 478

Subject Index 481