Contents

CHAPTER I. Motion of a particle

1.1 The free particle 1
1.2 Rectilinear motion in a field 3
1.3 Libration motion 9
1.4 The given force cannot be a function of acceleration . . . 11
1.5 The constrained particle (i) 12
1.6 The constrained particle (ii) 14
1.7 The constrained particle (iii) 15
1.8 Holonomic and non-holonomic systems 16
1.9 Two constraints 18

CHAPTER II. Dynamical systems

2.1 A simple example 20
2.2 The dynamical system 22
2.3 The catastatic system 24
2.4 The forces of constraint 24
2.5 The idea of a dynamical system 25

CHAPTER III. The first form of the fundamental equation

3.1 The fundamental equation 28
3.2 The conservation of momentum 29
3.3 The catastatic system and the first form of the equation of energy . 30
3.4 Conservative forces and the second form of the equation of energy . 31
3.5 The third form of the equation of energy 33
3.6 The conservation of energy 34
3.7 Hamilton's principle 34
3.8 The varied path 36
3.9 Continuous systems 37

CHAPTER IV. The second and third forms of the fundamental equation

4.1 The second form of the fundamental equation 40
4.2 The third form of the fundamental equation 41
4.3 Gauss's principle of Least Constraint 42
4.4 Applications of Gauss's principle 42
4.5 The physical significance of Gauss's principle 43

CHAPTER V. Lagrangian coordinates

5.1 The idea of Lagrangian coordinates 45
5.2 Some classical problems 47
5.3 The spherical pendulum 56
5.4 The problem of two bodies 59
5.5 Kepler's equation 62

5.6 Collision 63
5.7 Lagrangian coordinates for a holonomic system 64
5.8 Lagrangian coordinates for a non-holonomic system 65
5.9 Rolling bodies 67
5.10 Accessibility 69
5.11 The varied path in Hamilton's principle 69
5.12 Summary 71

CHAPTER VI. Lagrange's equations

6.1 The fourth form of the fundamental equation, Lagrangian coordinates 73
6.2 Lagrange's equations 75
6.3 Lagrange's equations deduced from Hamilton's Principle . . . 76
6.4 The form of the equations 78
6.5 Conservative systems and other systems with a potential function . 79
6.6 The Lagrangian function 81
6.7 Jacobi's integral 82
6.8 The explicit form of Jacobi's integral 83
6.9 An insidious fallacy 86
6.10 The generalized momentum-components 87
6.11 Ignorable coordinates 88
6.12 The invariance of Lagrange's equations 88

CHAPTER VII. The theory of rotations

7.1 Motion of a rigid body 90
7.2 Euler's theorem 90
7.3 The matrix $\mathbf{l}$ and the vector $\mathbf{T}$ 92
7.4 Generalization of Euler's theorem 94
7.5 Chasles's theorem 95
7.6 The rotation formula 95
7.7 Half-turns and reflexions 97
7.8 Quaternion form of the rotation formula 98
7.9 Composition of rotations 99
7.10 Angular velocity 102
7.11 The orientation of a rigid body, Euler's angles 103
7.12 The orientation of a rigid body, the angles φ_1, φ_2, φ_3 . . . 103
7.13 Rotations about the moving axes 104
7.14 Rotations about the fixed axes 104
7.15 Angular velocity found from $\mathbf{l}$ and $\dot{\mathbf{l}}$ 105
7.16 Components of angular velocity 106

CHAPTER VIII. First applications of Lagrange's equations

8.1 The differential equations 108
8.2 Formulae for acceleration in general orthogonal coordinates . . 109
8.3 Monkey and counterpoise 110
8.4 Kinetic energy of a rigid body 110
8.5 A problem of motion in two dimensions 111
8.6 The spinning top, the fundamental equations 113
8.7 The spinning top, another method 114

8.8 Gyroscopic forces 115
8.9 The spinning top, study of the motion 115
8.10 Numerical example 118
8.11 Rod in a rotating plane 119
8.12 The rolling penny 120

CHAPTER IX. The theory of vibrations

9.1 Oscillations about equilibrium 123
9.2 Theory of the transformation to normal coordinates 132
9.3 Application of the theory 137
9.4 Imposition of a constraint 140
9.5 Rayleigh's principle 141
9.6 Stability of a steady motion 143
9.7 Oscillations about steady motion 147
9.8 Foucault's gyroscope 150
9.9 The sleeping top 152
9.10 Forced oscillations 156

CHAPTER X. Further applications of Lagrange's equations

10.1 The ignoration of coordinates 159
10.2 Ignoration of a single coordinate 160
10.3 Gyroscopic stability 162
10.4 Explicit expression for R in the general case 164
10.5 The spinning top 165
10.6 Linear terms in L 166
10.7 Motion relative to a moving base 169
10.8 Motion of a particle near a given point on the Earth's surface . . 172
10.9 Foucault's pendulum 173
10.10 Projectiles 174
10.11 Rayleigh's dissipation function 179
10.12 Gyroscopic system with dissipation 181
10.13 Hamilton's equations 182
10.14 The equation of energy and the explicit form of H 185
10.15 The principal solid 187

CHAPTER XI. Variable mass

11.1 Particles with variable mass, the Lagrangian function . . . 190
11.2 The kinetic energy 191
11.3 The Hamiltonian function 192
11.4 The moving electron 192
11.5 Electron in an electromagnetic field 194

CHAPTER XII. The Gibbs-Appell equations

12.1 Non-holonomic systems 197
12.2 Quasi-coordinates 197
12.3 The fifth form of the fundamental equation 199
12.4 Determination of the acceleration 200
12.5 The Gibbs-Appell equations 201

CHAPTER XIII. Applications of the Gibbs-Appell equations

13.1 Particle moving in a plane 203
13.2 Analogue of König's theorem 204
13.3 Two-dimensional problems 204
13.4 Motion of a rigid body in space 205
13.5 Sphere on turntable 207
13.6 Sphere on a rotating inclined plane 209
13.7 Sphere rolling on a fixed surface 211
13.8 The spinning top 213
13.9 The rolling penny 214
13.10 Euler's equations 215
13.11 The free body, the case of axial symmetry 216
13.12 The free body, the general case 218
13.13 The free body, orientation 222
13.14 The theorems of Poinsot and Sylvester 223
13.15 Ellipsoid rolling on a rough horizontal plane 224
13.16 Stability of the spinning ellipsoid 225

CHAPTER XIV. Impulsive motion

14.1 The theory of impulses 228
14.2 Impulsive constraints 230
14.3 Impulsive motion of a system, the fundamental equation . . . 231
14.4 The catastatic system 232
14.5 The principle of Least Constraint for impulses 233
14.6 The catastatic system, the superposition theorem 234
14.7 The catastatic system, the six energy theorems 235
14.8 Lagrangian coordinates and quasi-coordinates 238
14.9 Lagrange's form of the impulse equations 241
14.10 The energy theorems reconsidered 242
14.11 Examples of impulsive motion 243
14.12 Impulsive motion of a continuous system 248

CHAPTER XV. The sixth form of the fundamental equation

15.1 The sixth form of the fundamental equation 253
15.2 Immediate deductions 253
15.3 The Routhian function 255
15.4 The theorem $\frac{d}{dt}(p_r\,\delta q_r) = \delta L$ 256
15.5 The principal function 258
15.6 Reflexions on the principal function 259
15.7 Proof that $\partial S/\partial t_0 = H_0$ 260
15.8 Properties of the principal function 261
15.9 Examples of the direct calculation of the principal function . . 265

CHAPTER XVI. The Hamilton-Jacobi theorem

16.1 Hamilton's partial differential equation 268
16.2 The Hamilton-Jacobi theorem, first proof 269

16.3 The equivalence theorem 271
16.4 The Hamilton-Jacobi theorem, second proof 275
16.5 Reflexions on the Hamilton-Jacobi theorem 275
16.6 Uniform field 277
16.7 The harmonic oscillator 279
16.8 Particle in a varying field At 281
16.9 Central orbit 282
16.10 The spherical pendulum 283
16.11 The spinning top 284
16.12 Rod in rotating plane 285
16.13 Electron under a central attraction 286
16.14 The Pfaffian form $p_r\,dq_r - H\,dt$ 288

CHAPTER XVII. Separable systems with two degrees of freedom

17.1 The idea of separability 291
17.2 Two degrees of freedom, conditions for separability 291
17.3 Study of the motion 294
17.4 Classification of the orbits 296
17.5 Stability 298
17.6 Application of the theory 299
17.7 Central attraction k/r^{n+1} 300
17.8 Central attraction k/r^5 302
17.9 Newtonian attraction and uniform field 306
17.10 Two fixed Newtonian centres 309
17.11 The bounded orbits 313
17.12 The equations of the orbits 316
17.13 The unbounded orbits 317
17.14 Systems that are separable in more than one way 318

CHAPTER XVIII. Separable systems with n degrees of freedom

18.1 Liouville's system 320
18.2 Stäckel's theorem 321
18.3 Discussion of the integrals 324
18.4 Further comments on Stäckel's theorem 325
18.5 Quasi-periodic motions 326
18.6 Angle variables 329
18.7 The standard cube 331
18.8 The constants I_r 332
18.9 Relations connecting q's and v's 334
18.10 Small oscillations 335
18.11 The spherical pendulum 337
18.12 The problem of two bodies 339
18.13 Interpretation of the α's and β's 341
18.14 Expression of r as a function of t 343
18.15 The angle variables 344
18.16 The constants I_r 344
18.17 Perturbation 347
18.18 Non-orthogonal and non-natural separable systems 348

CHAPTER XIX. Systems with one degree of freedom, motion near a singular point

19.1 The differential equations 349
19.2 Particle moving on a straight line 353
19.3 System with one degree of freedom 356
19.4 Motion in the neighbourhood of a singular point, the linear approximation 356
19.5 Stability of equilibrium, complete stability and instability . . 363
19.6 Motion in the neighbourhood of a singular point, the general theory 364
19.7 Motion near a node 365
19.8 Motion near a saddle point 368
19.9 Motion near a spiral point 371
19.10 Motion near a vortex point 372
19.11 Relation of the linear approximation to the general theory . . 376

CHAPTER XX. Systems with one degree of freedom, the cyclic characteristics

20.1 Index of a curve, and index of a singular point 379
20.2 The positive limiting set 381
20.3 Segment without contact 383
20.4 Segment without contact through a point of Λ 384
20.5 The structure of Λ 385
20.6 The Poincaré-Bendixson theorem 385
20.7 Application to a particular system 387
20.8 Existence of the limit cycle 389
20.9 Van der Pol's equation 392

CHAPTER XXI. Systems with n degrees of freedom, properties of the characteristics

21.1 Integrals of the system of differential equations 395
21.2 Transformation to new coordinates 399
21.3 The operator T_t 400
21.4 Solution in power series 401
21.5 A formula for $\mathbf{X}(\mathbf{x}) - \mathbf{X}(\boldsymbol{\alpha})$ 404
21.6 Integral-invariants 405
21.7 Integral-invariants of order m 408
21.8 Properties of the multipliers 410
21.9 Jacobi's last multiplier 411
21.10 The linear system 413
21.11 Stability of equilibrium 413
21.12 Discrete stability 415
21.13 Stability of transformations 417
21.14 Application to the differential equations 419
21.15 The Poincaré-Liapounov theorem 420
21.16 The critical case 423

CHAPTER XXII. Hamilton's equations

22.1 Hamilton's equations 427
22.2 Poisson brackets 428

22.3 Poisson's theorem 429
22.4 Use of a known integral 430
22.5 Poincaré's linear integral-invariant 433
22.6 Liouville's theorem 434
22.7 Poincaré's recurrence theorem 435
22.8 Examples of invariant regions 436
22.9 Ergodic theorems 437
22.10 Concrete illustrations 438
22.11 The set K_s 439
22.12 Proper segments 440
22.13 Proof of the ergodic theorem, first stage 441
22.14 Proof of the ergodic theorem, second stage 442
22.15 Metric indecomposability 443
22.16 Integrals of the equations of motion 446
22.17 A corollary to Liouville's theorem 446
22.18 The last multiplier 447

CHAPTER XXIII. Motion in the neighbourhood of a given motion, stability of motion

23.1 The variational equations 453
23.2 Solution of the variational equations 455
23.3 The case of constant coefficients 458
23.4 The case of periodic coefficients 461
23.5 Zero exponents 463
23.6 Variation from the Hamiltonian equations 465
23.7 Stability of trajectories (i) 467
23.8 Stability of trajectories (ii) 474
23.9 Stability of a periodic orbit 475
23.10 Forced oscillations 477

CHAPTER XXIV. Contact transformations

24.1 Contact transformations 484
24.2 Explicit formulae for a contact transformation 486
24.3 Other formulae 488
24.4 Extended point transformations and other homogeneous contact transformations 490
24.5 A special form of the equations of transformation, infinitesimal contact transformations 492
24.6 The extension of Liouville's theorem 492
24.7 The conditions for a contact transformation expressed in terms of Lagrange brackets 493
24.8 Relations between the two sets of derivatives 493
24.9 The conditions for a contact transformation expressed in terms of Poisson brackets 495
24.10 Relations between Lagrange brackets and Poisson brackets . . 495
24.11 Application to a contact transformation 496
24.12 Invariance of a Poisson bracket 496
24.13 Another form of the conditions for a contact transformation . . 497

24.14 Functions in involution 498
24.15 Some concrete examples 499

CHAPTER XXV. Transformation theory

25.1 The equations of motion after a contact transformation . . . 502
25.2 The variation of the elements 504
25.3 The variation of the elliptic elements 508
25.4 Other proofs of Jacobi's theorem 511
25.5 The constancy of Lagrange brackets 515
25.6 Infinitesimal contact transformations 516
25.7 Integrals in involution 517
25.8 Lie's theorem on involution systems 520
25.9 Integrals linear in the momenta 521
25.10 The case of a Hamiltonian function which is a homogeneous quadratic form 523

CHAPTER XXVI. Variation principles

26.1 Hamilton's principle 528
26.2 Livens's theorem 530
26.3 Minima and saddle points 531
26.4 Non-contemporaneous variations, Hölder's principle 533
26.5 Voss's principle 534
26.6 The generalization of Hamilton's principle 535
26.7 Change of the independent variable 536
26.8 Normal form for a system with two degrees of freedom . . . 537
26.9 Liouville's system 538
26.10 Conformal transformation 540

CHAPTER XXVII. The principle of Least Action

27.1 The variation of the Action 543
27.2 The principle of Least Action 544
27.3 Jacobi's form of the principle of Least Action 546
27.4 Whittaker's theorem 548
27.5 The ignoration of coordinates 550
27.6 The characteristic function 551
27.7 The configuration space 552
27.8 System with two degrees of freedom 553
27.9 Kelvin's theorem 554
27.10 Uniform field 556
27.11 Tait's problem, direct solution 558
27.12 Tait's problem, envelope theory 559

CHAPTER XXVIII. The restricted problem of three bodies

28.1 The problem of three bodies 562
28.2 The restricted problem, the equations of motion 563
28.3 Positions of equilibrium 564
28.4 Equilibrium points on AB 565
28.5 Equilibrium points not on AB 567

28.6 The surface $z = U$ 568
28.7 Motion near a point of equilibrium 569
28.8 Lunar theory 571

CHAPTER XXIX. The problem of three bodies

29.1 The classical integrals 573
29.2 The case of vanishing angular momentum 575
29.3 Lagrange's three particles 576
29.4 Fixed-shape solutions 578
29.5 Motion in a plane 580
29.6 Coordinates relative to A_3 582
29.7 Motion near the equilibrium solution 582
29.8 Reduction to the sixth order 585
29.9 Stability of Lagrange's three particles 587
29.10 Reduced form of the equations of motion 588
29.11 Lagrange's three particles reconsidered 590
29.12 Reduction to the eighth order 592
29.13 Impossibility of a triple collision 596
29.14 Motion in a plane, another method of reduction to the sixth order . 599
29.15 Equilibrium solutions 601

CHAPTER XXX. Periodic orbits

30.1 Periodic orbits 604
30.2 Periodic motion near a singular point 604
30.3 Reality conditions 607
30.4 Hamiltonian equations 608
30.5 Convergence 611
30.6 Lagrange's three particles 613
30.7 Systems involving a parameter 615
30.8 Application to the restricted problem of three bodies . . . 618
30.9 Fixed-point theory 621
30.10 Poincaré's ring theorem 621
30.11 Periodic orbits and the ring theorem 622
30.12 Proof of Poincaré's ring theorem 626

Notes 630
Bibliography 635
Index 639