

CONTENTS

PART I

I. THE CONCEPT OF PROBABILITY AND THE QUANTITATIVE PROBABILITY SCALE

A. INTRODUCTION, §§ 1–5	3
B. THE CONCEPT OF PROBABILITY, § 6.	5
1. Probability and frequency, §§ 7–9	5
2. Scale of probabilities, §§ 10–11	7
3. Law of addition of probabilities, §§ 12–14	8
4. Law of multiplication of probabilities, §§ 15–19	11
5. Connexion between laws of addition and of multiplication, § 20	14
6. Remark about scales, § 21.	15
7. Quantitative determination of probabilities in exceptional cases, §§ 22–23	16
C. SOME GENERAL CONCEPTS	17
1. The probability field and its exposures	17
(a) The probability field, § 24	17
(b) Exposure of a field, § 25	18
(c) Probability distributions, §§ 26–27	19
(d) Many-dimensional fields, §§ 28–29	21
(e) Regrouping of the elements of a field	23
(i) Contraction of a field, §§ 30–32	23
(ii) Combination of independent fields, § 33	24
2. Conditional probabilities	25
(a) Definition, §§ 34–38	25
(b) Statistics of radioactive decay, § 39	27
D. THE QUANTITATIVE SCALE OF PROBABILITIES	28
1. The Bernoulli distribution	28
(a) The Bernoulli formula, §§ 40–43	28
(b) Discussion of the Bernoulli distribution, §§ 44–46	31
(c) An analytic approximation of $B_p(N; k)$, §§ 47–50	33
(d) The error integral and the standard deviation, §§ 51–53	36
2. The quantitative scale of probabilities	39
(a) Estimate of the value of a probability, §§ 54–55	39
(b) Statistical margin of safety, §§ 56–59	40
(c) Poisson distribution and the probability distribution of rare events, §§ 60–61	43

II. EXPECTATION VALUES AND GENERATING FUNCTIONS	
A. EXPECTATION VALUES	45
1. The concept of expectation value, §§ 62–65	45
2. Expectation value of a function of k	48
(a) Expected square value, § 66	48
(b) Moments, § 67	48
(c) Expected value of a function of k , § 68	49
(d) Remark on functions that do not possess expected values, § 69	49
(e) Moments of negative order, §§ 70–71	50
3. The error of measurement	51
(a) The deviation δk , § 72	51
(b) Standard deviation, § 73	52
(c) Central moments of higher order, §§ 74–75	52
(d) Law of radioactive decay, § 76	53
4. Expectation values of various functions	55
(a) General relations, §§ 77–78	55
(b) Approximate expressions for distributions possessing a steep maximum, §§ 79–80	57
B. EXPECTATION VALUES OF FUNCTIONS OF SEVERAL VARIABLES	58
1. Two components	58
(a) Definitions, § 81	58
(b) Marginal distribution, §§ 82–84	58
(c) Expectation values of sums or products of functions, § 85	60
(d) The correlation coefficient, §§ 86–89	60
2. Several components, § 90	63
C. GENERATING FUNCTIONS	64
1. Definitions	64
(a) The function $\bar{G}(u)$, § 91	64
(b) Generating functions and moments, §§ 92–93	65
(c) Examples	66
(i) The Poisson distribution, § 94	66
(ii) The Bernoulli distribution, § 95	66
(d) Moment generating function, §§ 96–97	67
(e) Further generating functions, §§ 98–100	67
2. Many-dimensional fields, § 101	69
III. THE PROCESS OF FOLDING	
A. FOLDING OF TWO DISTRIBUTIONS	71
1. Definition, §§ 102–3	71
2. Folding with the help of generating functions, §§ 104–5.	73
B. THE POISSON DISTRIBUTION	74
1. Some aspects of the Poisson distribution, § 106	74

2. The distribution of random counts	75
(a) General consideration, §§ 107-8	75
(b) Postulate of rarity, § 109	76
3. Some properties of the Poisson distribution	77
(a) Limitation of validity of the Poisson distribution, §§ 110-12	77
(b) Counting with inefficient counters, § 113	78
(c) Distribution of samples taken from a series of counts, § 114	79
4. Generalized Poisson distribution, §§ 115-16	80
C. THE FOLDING OF SEVERAL DISTRIBUTIONS	82
1. Definitions, § 117	82
2. Folding of three distributions with the help of their generating functions, § 118	83
3. Folding of several distributions, §§ 119-20.	84
D. SOME APPLICATIONS	85
1. The simple Bernoulli distribution, §§ 121-2	85
2. Distribution of an average value	86
(a) Expected values and scatter, § 123	86
(b) A qualitative remark about the scatter of an average value, § 124	87
E. LOGARITHMIC MOMENTS	88
1. The logarithmic generating function, § 125	88
2. Logarithmic moments up to the third order, §§ 126-8	89
3. Moments of an average value, § 129	90
F. MANY-PARAMETER TRANSFORMS	91
1. Definitions, §§ 130-2	91
2. Many-dimensional moments, § 133	92
3. Folding of two-dimensional distributions, § 134	93
4. Generalized generating functions, § 135	94
5. Many-parameter Bernoulli distribution, §§ 136-8	94
IV. CONTINUOUS DISTRIBUTIONS	
A. THE CONCEPT OF CONTINUOUS DISTRIBUTION	98
1. Continuous distributions as approximations of discrete distributions	98
(a) The method of approximation, §§ 139-41	98
(b) Density and integral distribution, § 142	99
2. Essentially continuous distributions	101
(a) Distribution of decay times, § 143	101
(b) Probability density in general, § 144	101
3. Transformation of variables, § 145	102
4. Some properties of continuous distributions	104
(a) Expectation values	104
(i) Definition, § 146	104
(ii) Average value and expectation value, § 147	104

(b) Generating function of a continuous distribution	105
(i) The Laplace transform, § 148	105
(ii) Moments, § 149	106
(iii) Logarithmic moments, §§ 150-1	106
5. Particular distributions	107
(a) Gaussian distribution, §§ 152-3	107
(b) The exponential and Γ -distribution, §§ 154-5	108
B. FOLDING OF CONTINUOUS DISTRIBUTIONS	110
1. Definition, §§ 156-7	110
2. Folding with the help of the generating function	112
(a) Folding of two distributions, § 158	112
(b) Folding of several distributions, § 159	113
(c) Folding of Gaussian distributions, § 160	114
3. Construction of a distribution with the help of folding	115
(a) Distribution of rounding-off errors, § 161	115
(b) Statistics of radioactive decay times, §§ 162-3	116
(c) Statistics of counting times, § 164	117
(d) Connexion with the Poisson distribution, § 165	120
(e) Distribution of counts including effects of dead time, § 166	120
C. FURTHER TYPES OF CONTINUOUS DISTRIBUTION	122
(a) Definition of the distribution, §§ 167-9	122
(b) Expected values, § 170	124
(c) Times of arrival of particles, §§ 171-6	124
1. Mixed distributions and the problem of the δ -function	127
(a) The mixed distribution, §§ 177-8	127
(b) The δ^* -function, §§ 179-82	129
(c) The δ -function, § 183	131
(d) General consideration concerning discrete and continuous distributions, §§ 184-6	131
2. Distributions with several variables	133
(a) General considerations	133
(i) Expected values and generating function, § 187	133
(ii) Transformation of variables, §§ 188-9	134
D. THE GAUSSIAN DISTRIBUTION AND Γ-DISTRIBUTION	136
1. Gaussian distribution with two variables	136
(a) Moments, §§ 190-3	136
(b) Reduction of the Gaussian distribution of two variables	138
(i) Marginal distribution, § 194	138
(ii) Conditional probabilities, § 195	138
2. The N -component Gaussian distribution	139
(a) Normalization, §§ 196-8	139
(b) Moments of the N -component Gaussian distribution, §§ 199-200	142
(c) The inhomogeneous Gaussian distribution, §§ 201-3.	143

3. Connexion between the Gaussian distribution and the Γ -distribution	144
(a) The distribution of θ , §§ 204–5	144
(b) The margin of θ , § 206	145
4. Reductions of the N -component Gaussian distribution	147
(a) Marginal distribution, §§ 207–9	147
(b) Another connexion between the Gaussian and the Γ -distribution, § 210	150
(c) Conditional probabilities, § 211	150
(d) Distribution of the mean-square deviation from the average, § 212	152
(e) Approximation of a many-component Bernoulli distribution by a Gaussian distribution, §§ 213–15	153

V. HIGHER MOMENTS OF DISTRIBUTIONS

A. PARTICULAR DISTRIBUTIONS	156
1. Gaussian distribution, § 216	156
2. Γ -distribution, § 217	156
3. Poisson distribution, § 218	158
4. Bernoulli distribution, § 219	159
B. VARIOUS PROPERTIES OF HIGHER MOMENTS	161
1. Distributions with one variable	161
(a) Central moments and logarithmic moments	161
(i) Central moments, § 220	161
(ii) Logarithmic moments, § 221	161
(iii) Connexion between logarithmic and central moments, § 222	162
(b) Moments of distribution obtained by a large number of foldings	163
(i) The Gaussian approximation, § 223	163
(ii) An example	165
2. Higher moments of distributions with several variables	166
(a) Generating functions, § 224	166
(b) Many-dimensional moments, § 225	167
(c) Connexion between moments and central moments of an s -component distribution, § 226	168
(d) Connexion between logarithmic moments and central moments, § 227	169
3. Fourth-order central moments of a many-component Gaussian distribution	171
(a) Central moments, § 228	171
(b) Scatters of quadratic expressions	171
(c) A minimum property, § 229	172
C. DETERMINATION OF A DISTRIBUTION FUNCTION FROM ITS MOMENTS	172
1. The inverse Laplace transform, §§ 230–1	172

2. The saddle-point representation	173
(a) The method, §§ 232–3	173
(b) An example, § 234	174
3. Discussion of the properties of the saddle-point method, §§ 235–6	174
4. Estimate of the accuracy of the saddle-point method	176
(a) The generating function of $P^*(x)$, § 237	176
(b) Estimate of $l(v)$, § 238	177
(c) Second approximation to $P(x)$, § 239	177
(d) A general remark about the accuracy of the saddle-point method, §§ 240–2	178

PART II

VI. ESTIMATION OF PHYSICAL PARAMETERS FROM OBSERVATIONAL DATA

A. INTRODUCTORY REMARKS	185
1. The measured values, §§ 243–5	185
2. The concept of inverse probability, § 246	187
B. METHODS OF ESTIMATION OF ONE PARAMETER, § 247	188
1. Estimation of a parameter from one result of measurement	188
(a) The method of estimation, §§ 248–50	188
(b) The confidence parameter, § 251	191
(c) Graphical estimate of a parameter, § 252	192
(d) Estimate in terms of expected value and scatter, §§ 253–5	193
2. Estimation of one parameter from a series of measured results, § 256	196
(a) Reduction to the problem of estimation from one measured result, § 257	197
(b) Estimation of a parameter from an implicit relation, §§ 258–60	197
C. SIMULTANEOUS ESTIMATION OF SEVERAL UNKNOWN PARAMETERS	200
1. Two parameters	200
(a) The method of estimation, § 261	200
(b) Estimation with the help of two strips, §§ 262–3	202
(c) The confidence parameter in the case of the two-parameter estimate, § 264	204
(d) Estimation with the help of a rounded-off region, §§ 265–6	205
2. Estimation of parameters of a Gaussian distribution, §§ 267–71	206
3. Several parameters	211
(a) The method, §§ 272–3	211
(b) Gaussian distribution, § 274	212
(c) The general case, §§ 275–6	213
(i) Determination of a rounded-off region, § 277	214
(ii) Explicit expression for the rounded-off region, § 278	214

(d) Explicit determination of \mathcal{R}_s , § 279	216
(e) Moments of \bar{a}	217
(i) Expected value, § 280	217
(ii) The scatter of a , § 281	218
D. GENERAL REMARKS ON THE METHOD OF DETERMINING SEVERAL PARAMETERS	219
1. Remarks on the choice of \mathcal{R}_s , §§ 282–3	219
2. Superfluous parameters, §§ 284–5	221
(a) One superfluous parameter, § 286	222
(b) Methods of eliminating superfluous parameters, §§ 287–9	224
(c) Estimate of useful parameters, §§ 290–2	225
(d) Illustration with $s = 2, r = 1$, § 293	228
VII. OPTIMAL METHODS FOR THE ESTIMATION OF PARAMETERS	
1. Introductory remarks, § 294	230
A. ESTIMATION OF ONE UNKNOWN PARAMETER	230
1. Method of optimal estimation	230
(a) The minimum problem, §§ 295–6	230
(b) The scatter of the optimal measured value, § 297	233
(c) The maximum-likelihood principle, § 298	233
2. Applications of the maximum-likelihood method	234
(a) Gaussian distribution	234
(i) One-component distribution, § 299	234
(ii) Many-component Gaussian distributions with one unknown parameter, §§ 300–1	235
(b) Remark on the least-square method, § 302	237
(c) Poisson distribution, § 303	239
3. Limitation of the maximum-likelihood method	239
(a) Estimation of the length of a vector from its random projections on a plane, §§ 304–6	239
(b) Estimation of the length of a vector from components in random directions, §§ 307–8	242
B. MAXIMUM-LIKELIHOOD ESTIMATION OF SEVERAL SIMULTANEOUS PARAMETERS	244
1. Introductory remarks	244
(a) General remarks about optimal conditions, §§ 309–12	244
2. Optimum requirements, §§ 313–14	246
(a) The best choice of the functions $\mathcal{F}(a'; x)$, §§ 315–21	247
(b) Properties of the minimized region $\mathcal{R}_s^{(0)}$, § 322	251
3. Technique of estimating parameters with the maximum-likelihood method	253
(a) Notations, §§ 323–4	253

(b)	Exact estimation of parameters of the Poisson distribution, §§ 325-6	255
(c)	A practical method for the determination of parameters of the Poisson distribution, §§ 327-9	256
(d)	Solution of the maximum-likelihood equation by iteration, § 330	258
4.	Aspects of counting experiments	260
(a)	Measurement of interval length for fixed count, §§ 331-2	260
(b)	Statistics of the instants of arrival of signals, §§ 333-5	261
5.	Many-parameter Gaussian distribution, § 336	263
C.	FURTHER ASPECTS OF THE MAXIMUM-LIKELIHOOD METHOD	264
1.	Maximum likelihood and the generalized least-square method	264
(a)	Connexion in the case of a Gaussian distribution, §§ 337-9	264
(b)	Generalized least-square method, § 340-1	265
2.	Optimal determination of useful parameters, §§ 342-3	267
3.	Parameters with known values, §§ 344-5	268
VIII. CONSISTENCY TESTS		
1.	Introduction, §§ 346-7	271
A.	TESTS OF DISTRIBUTIONS WITH ONE VARIABLE	272
1.	Various types of tests	272
(a)	A test of the Gaussian distribution, §§ 348-9	272
(b)	Type of conclusions to be drawn from a test, § 350	272
(c)	Tests through upper and lower limits, §§ 351-2	273
(d)	Tests using moments, § 353	274
2.	Discrimination between two hypotheses	275
(a)	Non-overlapping intervals, §§ 354-5	275
(b)	Overlapping intervals, § 356	276
(c)	Interpolation between hypothetical distributions, §§ 357-9	277
(d)	Several hypotheses, §§ 360-3	280
B.	TESTS INVOLVING SEVERAL MEASURED VALUES, § 364	283
1.	Various types of test, § 365	283
(a)	Condition imposed on the extreme values of a series of results, §§ 366-8	284
(b)	The method of the histogram, §§ 369-71	288
2.	The χ^2 -test, §§ 372-3	289
(a)	Limitations of the χ^2 -test, §§ 374-5	291
(b)	Histograms with different subdivisions, §§ 376-84	292
(c)	Examples illustrating the efficiency of the χ^2 -test, §§ 385-6	297
(d)	χ^2 -test for a series of counter readings, § 387	299
(e)	Lower limit of detectable fluctuations, §§ 388-91	301

C. THE GENERALIZED THEORY OF TESTS	304
1. Testing function	304
(a) Examples, §§ 392-3	304
(b) Question of the 'best' test-function, § 394	306
(c) Tests with two functions, § 395	307
2. Test with regions in the x -space	308
(a) The method, § 396	308
(b) Considerations connected with two simultaneous tests	309
(i) Two regions, § 397	309
(ii) Confidence parameter of two simultaneous tests, § 398	310
(c) Another aspect of tests, §§ 399-402	312
(d) A remark on tests carried out with a number of test-functions, § 403	314
3. Methods of selecting suitable tests	314
(a) A paradox, §§ 404-5	314
(b) Remarks upon the methods of test, §§ 406-7	317
(c) Choice of consistency test, §§ 408-12	318
4. Consistency test by introduction of new parameters	320
(a) The method, § 413	320
(b) An example, §§ 414-15	321

PART III

IX. EXAMPLES FOR THE EVALUATION OF COUNTER RESULTS

A. MEASUREMENT OF INTENSITIES	327
1. Estimation of a constant intensity, §§ 416-17	327
2. Difference between two intensities	328
(a) Estimate of a difference, § 418	328
(b) Establishing a small difference, §§ 419-21	329
(c) Practical example for establishing a small difference, § 422	332
(d) Measurement of a large difference, §§ 423-4	334
(e) Determination of intensities above a background level, §§ 425-6	335
3. The ratio of two intensities	336
(a) Estimation of a ratio, § 427	336
(b) Determination of an absorption coefficient from two measured data, §§ 428-9	337
(c) Determination of an absorption coefficient in the presence of background, §§ 430-2	339
B. DETERMINATION OF PARAMETERS FROM A SERIES OF READINGS	343
1. Introduction, § 433	343

2. Determination of a decay constant	344
(a) The method, § 434	344
(b) Effect of cut-off, § 435	344
(c) Effect of background, § 436	345
(d) Recording with a multi-channel analyser	346
(i) Several channels, §§ 437–8	346
(ii) Two-channel discriminator, § 439	348
(iii) A practical example, § 440	349
3. Determination of the half-life of a radioactive source	350
(a) The method, § 441	350
(b) Determination of a decay time from counts in given intervals	352
(i) The method, § 442	352
(ii) Optimal subdivision of time, §§ 443–4	353
4. Determination of the parameters of an inhomogeneous source, §§ 445–7	356

X. CORRELATION PROBLEMS

A. SINGLE CORRELATION	359
1. The barometer effect	359
(a) Recording of timing of signals, §§ 448–9	359
(b) Recording in discrete intervals, § 450	361
(c) Regression coefficient and its significance in connexion with correlation, § 451	362
(d) Fluctuations exceeding the normal	363
(i) χ^2 -test, § 452	363
(ii) Correlation and χ^2 -test, § 453	364
2. Simultaneous recording with two arrangements, §§ 454–5	365
(a) Check of stability, §§ 456–8	365
(b) A measure of instability of an arrangement, § 459	369
(c) Estimation of the mean square amplitude of a true intensity fluctuation, § 460	370
B. MULTIPLE CORRELATIONS	370
1. Definitions, § 461	370
2. Correlation equations, §§ 462–4	372
3. An example for multiple correlation, § 465	374
(a) Single correlation, § 466	374
(b) Twofold correlation, § 467	376
(c) Triple correlation, § 468	376
(d) Correlation ellipse, § 469	377
(e) The mean scatter of the recorded intensity, § 470	378
(f) Fluctuation of corrected readings, §§ 471–2	378

XI. EVALUATION OF PARAMETERS FROM MEASUREMENTS UNDER VARIED CONDITIONS	
A. EXPERIMENTAL ABSORPTION LAWS	380
1. Determination of an absorption coefficient	380
(a) The maximum-likelihood relation, §§ 473-4	380
(b) Optimal choice of absorber thickness, §§ 475-6	381
(c) Remarks on the practical determination of an absorption coefficient, § 477	382
(d) A disadvantageous method, § 478	383
(e) A numerical example, § 479	384
2. Simultaneous determination of absorption coefficients of superimposed components	385
(a) The maximum-likelihood formalism, § 480	385
(b) Examples	387
(i) Estimation of an absorption coefficient (second method), § 481	387
(ii) A numerical example for two-component exponential absorption, §§ 482-3	388
B. ESTIMATION OF COEFFICIENTS OF LINEAR AGGREGATES	390
1. The maximum-likelihood approximation, §§ 484-7	390
2. An example, § 488	393
3. Optimal time distribution	395
(a) The method, §§ 489-90	395
(b) An example, §§ 491-3	396
C. DETERMINATION OF PARAMETERS FROM COMPLEX MEASUREMENTS	397
1. The problem, § 494	397
(a) Estimation of one parameter from a complex series of measurements, §§ 495-6	398
(b) Estimation of several parameters from a complex series of measurements, § 497	399
2. Determination of the neutron mean free path in a subcritical reactor	401
(a) Individual series of measurements, §§ 498-500	401
(b) Determination of the mean free path of neutrons, §§ 501-2	405
D. WEIGHTED MEAN VALUE	405
1. Mass of the μ -meson, §§ 503-6	405
2. Check of the experimental evidence of the formula of variation of mass of the electron with velocity, §§ 507-12	409
3. Analysis of the accuracy of the Balmer terms, §§ 513-14	412
XII. THE PROBLEM OF EVALUATION OF TRACKS IN EMULSION	416
A. SHORT SURVEY OF THE THEORY OF SCATTERING OF FAST PARTICLES, § 515	416
1. Definitions, §§ 516-20	416

2. Angle of scatter along a finite path, § 521	419
(a) Distribution of scattering angles, § 522	419
(b) Distribution of scattering angles (second method), §§ 523–5	420
(c) Distribution of scattering angles (third method), §§ 526–9	422
(d) The lateral scattering of a particle, §§ 530–4	424
3. Generalization of the third method	427
(a) One stochastic variable, §§ 535–6	427
(b) Simultaneous distributions of several stochastic variables, § 537	428
(c) Simultaneous distributions of angular and lateral scattering, § 538	429
(d) Distribution of the area between a track and its chord, § 539	430
(e) Distribution of sagittas	431
(i) Distribution of one sagitta, § 540	431
(ii) Simultaneous distribution of a number of sagittas, § 541	432
(iii) Distribution of a series of sagittas formed from equidistant points, § 542	433
B. MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS OF A TRACK IN AN EMULSION	434
1. The one-parameter problem, § 543	434
(a) The exact estimate, §§ 544–5	434
(b) Practical expressions for the estimation of α , §§ 546–7	435
(c) Estimate using the average of the absolute values of the sagittas, § 548	437
2. Estimate of the Coulomb scattering parameter in the presence of background noise, § 549	437
(a) Formulation of the problem, §§ 550–2	438
(b) The maximum-likelihood relation, §§ 553–4	439
(c) The minimum scatter, §§ 555–7	440
(d) Error of $\bar{\alpha}^{(1)}$ in case of a known noise level $\alpha^{(2)}$, § 558	443
3. Practical estimates of $\alpha^{(\lambda)}$	443
(a) Numerical methods, §§ 559–61	443
(b) Optimal subdivision of a track, §§ 562–4	449
(c) A numerical example, §§ 565–6	451
C. METHODS OF ESTIMATION CONSIDERING DISTORTIONS OF THE EMULSION	453
1. Effects of distortions of the emulsion	453
(a) Long-range distortions, §§ 567–70	453
2. Spurious scattering, § 571	455
(a) Description of the effect, § 572	455
(b) Suggestions as to the mechanism of spurious scattering, § 573	456
(c) Statistical treatment of ‘random-walk’ distortions, § 574	459
(d) Hypothesis as to the nature of spurious scattering, § 575	460
3. Maximum-likelihood estimate of three parameters	460
(a) The exact method, §§ 576–7	460

CONTENTS

xxi

(b) Practical method of estimation in the presence of spurious scattering, § 578	463
4. Deviations of Gaussian distributions, § 579	466
D. A PROBLEM CONNECTED WITH THE SCANNING OF EMULSIONS, §§ 580-4	466
APPENDIX I, §§ 585-6	472
APPENDIX II	476
AUTHOR AND SUBJECT INDEX	479