

## CONTENTS

	Preface	ix
1	Some basic mathematics	1
1.1	The space $R^n$ and its topology	1
1.2	Mappings	5
1.3	Real analysis	9
1.4	Group theory	1 1
1.5	Linear algebra	13
1.6	The algebra of square matrices	. 16
1.7	Bibliography	20
2	Differentiable manifolds and tensors	23
2.1	Definition of a manifold	•23
2.2	The sphere as a manifold	26
2.3	Other examples of manifolds	28
2.4	Global considerations	29
2.5	Curves	30
2.6	Functions on M	30
2.7	Vectors and vector fields	31
2.8	Basis vectors and basis vector fields	34
2.9	Fiber bundles	35
2.10	Examples of fiber bundles	37
2.11	A deeper look at fiber bundles	38
2.12	Vector fields and integral curves	42
2.13	Exponentiation of the operator d/dh	43
2.14	Lie brackets and noncoordinate bases	43
2.15	When is a basis a coordinate basis?	47
2.16	One-forms	49
2.17	Examples of one-forms	50
2.18	The Dirac delta function	51
2.19	The gradient and the pictorial representation of a one-form	52
2.20	Basis one-forms and components of one-forms	55
2.21	Index notation	56

	Contents	vi
2.22	Tensors and tensor fields	57
2.23	Examples of tensors	58
2.24	Components of tensors and the outer product	59
2.25	Contraction	59
2.26	Basis transformations	60
2.27	Tensor operations on components	63
2.28	Functions and scalars	64
2.29	The metric tensor on a vector space	64
2.30	The metric tensor field on a manifold	68
2.31	Special relativity	70
2.32	Bibliography	71
3	Lie derivatives and Lie groups	73
3.1	Introduction: how a vector field maps a manifold into itself	73
3.2	Lie dragging a function	74
3.3	Lie dragging a vector field	74
3.4	Lie derivatives	76
3.5	Lie derivative of a one-form	78
3.6	Submanifolds	79
3.7	Frobenius' theorem (vector field version)	81
	Proof of Frobenius' theorem	83
3.9	An example: the generators of $S^2$	85
3.10	Invariance	86
	Killing vector fields	. 88
3.12	Killing vectors and conserved quantities in particle dynamics	89
3.13	Axial symmetry	89
3.14	Abstract Lie groups	92
3.15	Examples of Lie groups	95
3.16	Lie algebras and their groups	101
	Realizations and representations	105
3.18	Spherical symmetry, spherical harmonics and representations	100
	of the rotation group	108
3.19	Bibliography	112
4	Differential forms	113
_	The algebra and integral calculus of forms	113
	Definition of volume — the geometrical role of differential	
	forms	113
4.2	Notation and definitions for antisymmetric tensors	115
	Differential forms	117
(S) 74970 (O)	Manipulating differential forms	119
	Restriction of forms	120
	Fields of forms	120

	Contents	vii
4.7	Handedness and orientability	121
4.8	Volumes and integration on oriented manifolds	121
4.9	N-vectors, duals, and the symbol $\epsilon_{ijk}$	125
4.10	Tensor densities	128
4.11	Generalized Kronecker deltas	130
4.12	Determinants and $\epsilon_{ijk}$	131
	Metric volume elements	132
В	The differential calculus of forms and its applications	134
4.14	The exterior derivative	134
4.15	Notation for derivatives	135
4.16	Familiar examples of exterior differentiation	136
4.17	Integrability conditions for partial differential equations	137
4.18	Exact forms	138
4.19	Proof of the local exactness of closed forms	140
4.20	Lie derivatives of forms	142
4.21	Lie derivatives and exterior derivatives commute	143
4.22	Stokes' theorem	144
4.23	Gauss' theorem and the definition of divergence	147
4.24	A glance at cohomology theory	150
4.25	Differential forms and differential equations	152
4.26	Frobenius' theorem (differential forms version)	154
4.27	Proof of the equivalence of the two versions of Frobenius'	
	theorem	157
4.28	Conservation laws	158
4.29	Vector spherical harmonics	160
4.30	Bibliography	161
5	Applications in physics	163
Α	Thermodynamics	163
5.1	Simple systems	163
5.2	Maxwell and other mathematical identities	164
5.3	Composite thermodynamic systems: Caratheodory's theorem	165
В	Hamiltonian mechanics	167
5.4	Hamiltonian vector fields	167
5.5	Canonical transformations	168
5.6	Map between vectors and one-forms provided by $\widetilde{\omega}$	169
5.7	Poisson bracket '	170
5.8	Many-particle systems: symplectic forms	170
5.9	Linear dynamical systems: the symplectic inner product and	
	conserved quantities	171
5.10	Fiber bundle structure of the Hamiltonian equations	174
C	Electromagnetism	175
5.11	Rewriting Maxwell's equations using differential forms	175

	Contents	viii
5.12	Charge and topology	179
5.13	The vector potential	180
5.14	Plane waves: a simple example	181
D	Dynamics of a perfect fluid	181
5.15	Role of Lie derivatives	181
5.16	The comoving time-derivative	182
5.17	Equation of motion	183
5.18	Conservation of vorticity	184
E	Cosmology	186
5.19	The cosmological principle	186
5.20	Lie algebra of maximal symmetry	190
5.21	The metric of a spherically symmetric three-space	192
5.22	Construction of the six Killing vectors	195
5.23	Open, closed, and flat universes	197
5.24	Bibliography	199
6	Connections for Riemannian manifolds and gauge theories	201
6.1	Introduction	201
6.2	Parallelism on curved surfaces	201
6.3	The covariant derivative	203
6.4	Components: covariant derivatives of the basis	205
6.5	Torsion	207
6.6	Geodesics	208
6.7	Normal coordinates	210
6.8	Riemann tensor	210
6.9	Geometric interpretation of the Riemann tensor	212
6.10	Flat spaces	214
6.11	Compatibility of the connection with volume-measure or the	3 - 2 - 2 - 2 - 2 - 2 - 2 - 2 - 2 - 2 -
	metric	215
	Metric connections	216
	The affine connection and the equivalence principle	218
6.14	Connections and gauge theories: the example of	
ند د د	electromagnetism	219
6.15	Bibliography	222
	Appendix: solutions and hints for selected exercises	224
	Notation	244
	Index	246

