



# CONTENTS

Preface

vii

## CHAPTER 1 : DYNAMIC SYSTEMS AND RECURRENCES. GENERALITIES.

1.1.	Continuous dynamic systems and discrete dynamic systems.	1
1.2.	Birkhoff classification of dynamic motions. Impredictibility of chaotic motions.	3
1.3.	Some considerations about recurrences.	6
1.4.	Considerations about stability. Some definitions.	9
1.5.	The Poincaré's method of surface of section.	11
1.6.	Diffeomorphism and endomorphism. The Hadamard's theorem. The two classes of endomorphisms.	14
1.7.	Imbedding of an endomorphism into a diffeomorphism with a higher dimension.	16
1.8.	The Valiron's results.	17
1.9.	Consequences of the Valiron's results.	20
1.10.	Autonomous recurrences and the Schröder's equation.	21
1.10.1.	Generalities.	21
1.10.2.	Basin boundary and the Schröder equation.	22
1.10.3.	One-dimensional recurrences and chaotic behaviour. Generalization of the Chebyshev's polynomials.	24
1.10.4.	Generalization to $m$ -dimensional recurrences $m > 1$ .	29
1.11.	The problem of the fractional iterates.	31

## CHAPTER 2 : SOME PROPERTIES OF ONE-DIMENSIONAL RECURRENCES (maps).

	Introduction.	33
2.1.	Autonomous linear case.	34
2.2.	Autonomous non linear case. The singularities.	36
2.3.	Autonomous non linear case. Stability of fixed point and cycles.	40
2.3.1.	Recurrence defined by a smooth function.	40
2.3.2.	The function defining the recurrence is not a smooth one.	45
2.4.	Autonomous non linear case. Local bifurcations.	47
2.4.1.	Recurrence defined by a smooth function.	48
2.4.2.	Recurrence defined by a continuous but not smooth function.	54



2.5.	Bifurcations of piecewise continuous recurrences.	56
2.5.1.	Piecewise linear recurrences. Boxes in files bifurcations structure.	56
2.5.2.	Piecewise continuous recurrences.	62
2.6.	Properties of invertible one-dimensional recurrences.	62
2.7.	Properties of non-invertible one-dimensional recurrences.	63
2.7.1.	The critical points.	63
2.7.2.	Fatou's theorems about critical points.	64
2.7.3.	Construction of $T^k$ from the map $T$ .	65
2.7.4.	Cycles of one-dimensional endomorphisms.	68
2.7.5.	Absorbing segment.	69
2.8.	Domain of attraction (Basin) of an attractive fixed point. Some global bifurcations.	70
2.8.1.	All in one block basin with a boundary made up of two points.	70
2.8.2.	Basin made up of a finite, or an infinite number of disjointed parts with only one accumulation point. First example of global bifurcation.	71
2.8.3.	Complex basin of the first type. Second example of global bifurcation. Chaotic transient.	74
2.8.4.	Complex basin of second type. Chaotic (or fuzzy) basin boundary. Third example of a global bifurcation.	77
2.8.5.	Some remarks.	78
2.9.	Invertible map of the circle onto itself.	80
2.9.1.	Diffeomorphism of the circle.	80
2.9.2.	Generalizations.	84
2.9.3.	Boxes in files bifurcations structure and Farey sequences.	85
2.9.4.	Boxes in files bifurcations structure and the phenomenon of phase intermittency.	85
CHAPTER 3 : MYRBERG'S RESULTS ON THE ONE-DIMENSIONAL QUADRATIC RECURRENCES. THEIR CONSEQUENCES.		
3.1.	Introduction.	87
3.2.	Some general properties.	90
3.2.1.	First set of Myrberg's results.	90
3.2.2.	Number of all the possible cycles of order $k$ .	93
3.2.3.	Number of the bifurcations giving rise to the cycles of order $k$ .	94
3.2.4.	Properties related to $\lambda = \lambda_1^*$ .	97
3.2.5.	Properties related to $\lambda > \lambda_1^*$ .	102



3.3.	The Myrberg's results.	103
3.3.1.	The Myrberg's characterization of a cycle point.	103
3.3.2.	The Myrberg's rotation sequence.	105
3.3.3.	The Myrberg's ordering law. Consequences.	106
3.3.4.	The Myrberg's ordering law and its relation to the notion of invariant coordinate and kneading invariant.	109
3.3.5.	Fundamental bifurcations and Myrberg's singular parameter values of the first type.	110
3.3.6.	Variant of the singular parameter values of the first type.	113
3.3.7.	Consequence of the Myrberg ordering law : first numbering of the cycles of the same order. Non-embedded representation.	113
3.3.8.	Myrberg singular parameter values of the second type.	114
3.3.9.	Myrberg's singular parameter values of the third type.	116
3.4.	Convergence of rotation sequences toward a singular value.	119
3.4.1.	Convergence toward a singular value of the first type.	119
3.4.2.	Convergence toward a singular value of the second type.	120
3.4.3.	Convergence toward a singular value of the third type.	124
3.4.4.	Remark.	126
3.5.	The Algorithms giving the set of all the ordered cycles of order k.	126
3.5.1.	Notations and definitions.	126
3.5.2.	First algorithm.	127
3.5.3.	Second algorithm.	127
3.6.	Adjoint, and self adjoint, binary words of the cycles of unimodal one dimensional endomorphisms. Symmetry.	130
3.7.	The complete non-embedded representation of a cycle.	131

#### CHAPTER 4 : THE BOX-WITHIN-A-BOX BIFURCATIONS STRUCTURE AND ITS CONSEQUENCES.

4.1.	Introduction.	133
4.2.	Characterization of a cycle by a decimal rotation sequence.	135
4.2.1.	Definition of the decimal rotation sequence.	135
4.2.2.	Deduction of the binary sequence from the decimal one.	135
4.2.3.	Deduction of the decimal rotation sequence from the binary one.	136



4.2.4.	Necessary and sufficient condition for a permutation of the $k$ first integers to be a decimal rotation sequence of a map defined by a function having a single extremum.	136
4.2.5.	The subset of the rotation sequences directly related to the Poincaré's rotation numbers.	138
4.3.	Decomposable and indecomposable rotation sequences $[u]$ .	142
4.3.1.	Definition and examples.	142
4.3.2.	Second numbering of the cycles. Embedded representation.	144
4.3.3.	Composition of indecomposable rotation sequences.	146
4.4.	The box-within-a-box bifurcations structure (fractal structure).	149
4.4.1.	Simple (or non-embedded) boxes of the structure.	149
4.4.2.	Embedded boxes of first kind.	151
4.4.3.	Embedded boxes of second kind.	153
4.4.4.	Accumulation points of the boxes $(f(x, \lambda) \equiv x^2 - \lambda)$ .	156
4.5.	Resulting properties on the $x$ -axis.	161
4.5.1.	The parameter value $\lambda$ is not singular in the sense of Myrberg.	161
4.5.2.	The parameter value $\lambda$ is singular of the first type.	162
4.5.3.	The parameter value $\lambda$ is singular of the second type.	162
4.5.4.	The parameter value $\lambda$ is singular of the third type. Phenomenon of intermittency.	164
4.6.	Invariant measures associated with singular parameters values of the second type.	166
4.7.	Topological entropy.	174
4.8.	Some generalizations.	176
4.8.1.	Recurrence defined by a function with only one extremum.	176
4.8.2.	Recurrence defined by a function with two extrema.	178
4.9.	Recurrence defined by a continuous piecewise linear function with only one extremum.	179
CHAPTER 5 : SOME PROPERTIES OF TWO-DIMENSIONAL RECURRENCES		
5.1.	Introduction.	184
5.2.	Autonomous linear case.	188
5.2.1.	General solution.	188
5.2.2.	Invariant curves when the multipliers are real and different.	190



5.2.3.	Invariant curves when the multipliers are complex.	191
5.2.4.	Case of two real multipliers with equal moduli.	194
5.2.5.	Degenerate singular points.	194
5.3.	Autonomous non linear case. The invariant curves.	195
5.4.	Autonomous non linear case. Homoclinic and heteroclinic points.	201
5.5.	Critical case $ S_1  = 1,  S_2  \neq 1$ .	202
5.5.1.	Particular case. The map has a whole curve made up of fixed points, or of points of order two cycles.	202
5.5.2.	General case.	203
5.6.	Bifurcations by passing through the multiplier $S_1 = \pm 1$ .	207
5.6.1.	Passing through the particular critical case of § 5.5.1.	207
5.6.2.	General case with one multiplier $S_1 = +1$ .	211
5.6.3.	General case with one multiplier $S_1 = -1$ .	214
5.7.	Critical case with multipliers $S_{1,2} = \exp(\pm j \phi)$ , $j = \sqrt{-1}$ . The non-exceptional case.	215
5.8.	The exceptional case with $\phi = 2k\pi/q$ , $q = \text{odd integer}$ .	220
5.8.1.	First situation.	220
5.8.2.	Second situation.	221
5.9.	The exceptional case with $\phi = 2k\pi/q$ , $q = \text{even integer}$ .	224
5.9.1.	First situation.	225
5.9.2.	Second situation.	229
5.10.	Critical cases with multipliers $S = \exp(\pm j \phi)$ , $\phi = 0$ , or $\pi$ .	231
5.11.	Bifurcation by crossing through the critical case $S = \exp(\pm j \phi)$ .	234
5.11.1.	Crossing through a non-exceptional case.	234
5.11.2.	Crossing through an exceptional case.	239
5.12.	Critical case with two multipliers $S_1 = S_2 = +1$ when the linear approximation matrix is not reducible to a diagonal form.	239
5.13.	Bifurcation by crossing the critical case of the § 5.12.	245
5.13.1.	Crossing through critical cases with real principal invariant curves.	247
5.13.2.	Crossing through critical cases without real principal invariant curves.	250
5.14.	Birth of invariant closed curves from a linear conservative case perturbed by small non linear dissipative terms.	251



5.15. Birth of invariant closed curves in some other cases of global bifurcations.	255
5.16. First example of the bifurcations of the two precedent paragraphs.	258
5.17. Second example. Global bifurcations in presence of chaotic behaviour.	266

## CHAPTER 6 : TWO-DIMENSIONAL DIFFEOMORPHISMS AND THE FOLIATED BOX-WITHIN-A-BOX BIFURCATIONS STRUCTURE.

6.1. Introduction.	280
6.2. Behaviour of the diffeomorphism for small values of $ b $ .	286
6.2.1. Degenerate $\omega$ -invariant curves of $T_b$ for $b = 0$ , and invariant curves for $b \neq 0$ .	286
6.2.2. Consequence : "oscillating" basins.	289
6.2.3. $\alpha$ -invariant curves of the map $T_b$ , for small values of $ b $ .	291
6.2.4. Complex basins : chaotic transients, fuzzy (or chaotic) basins boundary.	296
6.3. Behaviour of the diffeomorphism in the conservative case.	298
6.3.1. Quadratic case $b = -1$ . Properties of the phase plane.	298
6.3.2. Quadratic case $b = -1$ . Local bifurcations.	304
6.3.3. Non quadratic case $b = -1$ .	309
6.3.4. Behaviour for $b = +1$ .	310
6.4. Behaviour of the diffeomorphism in the almost conservative case.	311
6.5. Quadratic case. First set of properties of the parameter plane.	313
6.6. The Smale horseshoe and the two-dimensional quadratic diffeomorphism. Representation of the cycles.	317
6.6.1. The Smale horseshoe.	317
6.6.2. Representation of the cycles. Symmetry.	319
6.7. Quadratic case. Box-within-a-box foliated bifurcation structure.	326
6.7.1. Origin of the foliation.	326
6.7.2. Homoclinic and heteroclinic situations. Morse-Smale area.	334
6.8. Some rules of formation of cycles represented in the symbolic phase plane.	341
6.9. Singularities of the fractal foliated bifurcations structure. Communications between sheets of this structure.	349
6.9.1. Generalities.	349
6.9.2. The cross-road area.	351



6.9.3. The saddle area.	358
6.9.4. The spring area. Local and global aspects.	361
6.10. Contracted representation of the parameters plane. Consequence.	373
6.11. Symbolic dynamics of communications in the fractal foliated bifurcations structure. Connections chain.	375
6.11.1. Notions of connections chain and communi- cations cell.	375
6.11.2. Connections chain and communications cell of simple situations.	378
6.11.3. Connections chain and communications cell of more complex situations.	388
6.11.4. Organization of connections chain belonging to a same list $E_k$ ( $b < 0$ ).	391
6.12. Bifurcations structure of two-dimensional cubic diffeomorphism.	395
6.12.1. Generalities.	395
6.12.2. Description of the chains of bifurcations (6.41).	397
Appendix A - Poincaré's indexes $ B3 $ .	407
Appendix B - Equivalence between the canonical form of a two-dimensional recurrence and the reduced Cigala form.	411
Appendix C - Critical case with two pairs of complex multipliers in a four-dimensional non-linear recurrence.	413
Appendix D - Particular case of a two-dimensional recurrence with real variables reduced to a one-dimensional recurrence with complex variable.	421
Appendix E - The problem of structural stability in the chaotic dynamic situations.	424
References	427
Index	445