



# Contents

*Preface*

ix

## 1. Probability Distributions and Densities

1.1	Definition of Probability	1
1.2	Formalizing the Definition	3
1.3	Measure and Lebesgue Integration	5
1.4	Distribution Function	7
1.5	The Probability Density Function	8
1.6	A Simple Example of the Distribution and Density Functions	10
1.7	Expected Values	11
1.8	Joint Distribution Functions	12
1.9	The Joint Density Function	13
1.10	Expected Values	14
1.11	Conditional Probabilities	15
1.12	Conditional Expectations	16
1.13	Statistical Independence and Lack of Correlation	16
1.14	Change of Variable	17

## 2. Moments, Characteristic Functions, and the Gaussian Distribution

2.1	Moments Defined	19
2.2	Significance of the Moments	21
2.3	The Correlation Matrix and Principal Axes	24
2.4	The Characteristic Function	25
2.5	Properties of the Characteristic Function	25
2.6	Two Simple Examples	27
2.7	The Central-Limit Theorem for Independent Variables	30
2.8	The Gaussian Distribution from a More Physical Point of View	33
2.9	Moments of a Gaussian Distribution	37
2.10	The Jointly Normal Distribution	38
2.11	Cumulants	39
2.12	The Gram–Charlier Expansion	39

### 3. Random Functions

3.1	Generalities: Multipoint Characteristic Functions	43
3.2	Statistics for Derivatives and Integrals	44
3.3	Processes and Characteristic Functionals: The Gaussian Process	47
3.4	Limit Processes of Random Functions	50
3.5	The Representation Problem	54
3.6	Finite Total Energy and Characteristic Eddies	57
3.7	Calculation of the Characteristic Eddies	59
3.8	Rate of Convergence of the Series of Eigenfunctions	60
3.9	Stationarity and the Ergodic Problem	62
3.10	Autocorrelations of Stationary Processes and Their Properties	68
3.11	Estimation by Time Averages	71
3.12	The Representation Problem for Stationary Processes: Spectra	76
3.13	Estimation by Time Averages with Zero Integral Scale	79
3.14	Another Type of Representation Theorem for Stationary Processes: Characteristic Eddies	80
3.15	Alternate Approaches to Harmonic Decomposition for Stationary Processes	82
3.16	A Central-Limit Theorem for Random Functions	84

### 4. Random Processes in More Dimensions

4.1	Multidimensional Vector Fields of Finite Energy	95
4.2	Homogeneity, Averaging, and Ergodicity in Several Dimensions	96
4.3	The Homogeneous Scalar Field: One-Dimensional Spectra	98
4.4	The Homogeneous Scalar Field: The Three-Dimensional Spectrum	100
4.5	The Homogeneous Scalar Field: Consequences of Isotropy	100
4.6	The Homogeneous Scalar Field: General Form of the Spectra	102
4.7	The Solenoidal Homogeneous Vector Field: Implications of Incompressibility	104
4.8	The Solenoidal Homogeneous Vector Field: One-Dimensional Spectra	107
4.9	The Solenoidal Homogeneous Vector Field: The Three-Dimensional Spectrum	107
4.10	The Solenoidal Homogeneous Vector Field: Consequences of Isotropy	108
4.11	The Solenoidal Homogeneous Vector Field: General Form of the Spectra	111
4.12	Characteristic Eddies for a Homogeneous Vector Field	113
4.13	Incompletely Homogeneous Fields: Co- and Quadrature Spectra and Coherence	114
4.14	Characteristic Eddies for an Incompletely Homogeneous Field	117
4.15	Multiple-Valued Functions	121
4.16	Distribution of Solutions for an Algebraic Equation	130

### Appendix 1. Fourier Transforms

A1.1	Fourier Transforms of Well-Behaved Functions	137
A1.2	The Inverse Transform	138
A1.3	The Convolution	139

A1.4	Symmetry Properties	140
A1.5	Parseval's Relation	141
A1.6	Relations among Derivatives	142
A1.7	Shift of Variables	142
A1.8	Multiple Variables	143
<b>Appendix 2. Tensors</b>		
A2.1	Transformation Properties: Co- and Contravariant Indices	145
A2.2	The Metric Tensor: Changing Indices	147
A2.3	Cartesian Systems, Numerical Tensors, and Tensor Densities	149
A2.4	Differentiation	152
A2.5	Eigenvalues and Eigenvectors: Representations	153
A2.6	Principal Invariants, The Cayley-Hamilton Theorem, and Inverses	156
<b>Appendix 3. Theory of Generalized Functions</b>		
A3.1	Generalities	159
A3.2	Linear Continuous Functionals	160
A3.3	Addition and Multiplication by a Constant and by a Function	161
A3.4	Convergence of Sequences of Generalized Functions	162
A3.5	Differentiation and Integration of Generalized Functions	162
A3.6	Support of a Generalized Function	164
A3.7	Direct Product of Generalized Functions	164
A3.8	Convolutions of Generalized Functions	164
A3.9	Fourier Transforms of Generalized Functions	166
A3.10	Several Variables	167
A3.11	Effect of a Shift of Variables	167
A3.12	Asymptotic Behavior of Generalized Functions	168
A3.13	Fourier Transforms of Generalized Functions Defined on the Space of Bounded, Infinitely Differentiable Functions	172
A3.14	The Fourier Transform of the Convolution	174
A3.15	Behavior of the Fourier Transform	175
A3.16	The Kernel Theorem	176
A3.17	Representations of Generalized Functions of Finite Total Energy	176
<b>Appendix 4. Invariant Theory, Isotropy, and Axisymmetry</b>		
A4.1	Invariance under Transformation Groups	179
A4.2	Independent Invariants of Tensors of Various Orders	180
A4.3	Representations of Tensor Functions—A General Method	182
A4.4	Tensor Constants and Functions of a Scalar	183
A4.5	Tensor Functions of a Vector	183
A4.6	Tensor Functions of a Tensor of Second Rank	184
<b>References</b>		187
<i>Index</i>		191