

CONTENTS

Preface	v
CHAPTER I. Introduction to Partial Differential Equations	1
1. Introduction	1
2. The one-dimensional wave equation	3
3. Method of separation of variables	6
4. The two-dimensional wave equation	12
5. Three-dimensional wave equation	15
6. The wave equation in plane and cylindrical polar coordinates	17
a. Plane polars	17
b. Cylindrical polars	18
7. The wave equation in spherical polar coordinates	19
8. Laplace's equation in two dimensions	23
a. Cartesian coordinates	23
b. Polar coordinates	25
9. Laplace's equation in three dimensions	29
10. The diffusion or heat flow equation	31
10.1. Neutron diffusion	36
11. A fourth order partial differential equation	38
12. The bending of an elastic plate — the biharmonic equation .	40
13. Characteristics	44
13.1. Cauchy's problem	44
13.2. Reduction of (13.1.1) to the standard form	47
a. Hyperbolic equation	48
b. Elliptic equation	48
c. Parabolic equation	49
13.3. Riemann's method of solution of (13.1.1)	50
a. Hyperbolic equation: $H^2 > AB$	52
b. Transformation to the (X, Y) -plane	58
13.4. Numerical integration of hyperbolic differential equations	59
Problems	63
General References	68
CHAPTER II. Ordinary Differential Equations: Frobenius' and Other Methods of Solution	69
1. Introduction	69
2. Solution in series by the method of Frobenius	70
3. Bessel's equation	75
4. Legendre's equation	82
5. Hypergeometric equation	86

CONTENTS

6. Series solution about a point other than the origin	88
6.1. The transformation $x = (1 - \xi)/2$	88
a. Tschebyscheff polynomials of the first kind	89
b. Jacobi polynomials	90
c. Tschebyscheff polynomials of the second kind	91
7. Series solution in descending powers of x	91
8. Confluent hypergeometric equation	96
8.1. Laguerre polynomials	97
a. Associated Laguerre polynomials, $L_n^{(k)}(x)$	99
b. Sonine polynomials	101
8.2. Hermite polynomials	101
9. Asymptotic or semi-convergent series	102
10. Change of dependent variable	104
11. Change of the independent variable	108
12. Exact equations	109
13. The inhomogeneous linear equation	112
14. Perturbation theory for non-linear differential equations .	115
14.1. The perturbation method	115
14.2. Periodic solutions	118
Problems	122
General References	126
CHAPTER III. Bessel and Legendre Functions	127
1. Definition of special functions	127
2. $J_n(x)$, the Bessel function of the first kind of order n . .	128
2.1. Recurrence relations: $J_n(x)$	131
3. Bessel function of the second kind of order n , $Y_n(x)$. .	133
4. Equations reducible to Bessel's equation	138
5. Applications	139
6. Modified Bessel functions: $I_n(x)$, $K_n(x)$	143
6.1. Recurrence relations for $I_n(x)$ and $K_n(x)$	145
6.2. Equations reducible to Bessel's modified equation . .	147
6.3. Bessel functions of the third kind (Hankel functions) .	147
7. Illustrations involving modified Bessel functions	148
8. Orthogonal properties	154
8.1. Expansion of $f(x)$ in terms of $J_n(\xi; x)$	156
8.2. $J_n(x)$ as an integral (where n is zero or an integer) . .	159
8.3. Other important integrals	161
9. Integrals involving the modified Bessel functions	171
10. Zeros of the Bessel functions	174
11. A generating function for the Legendre polynomials . . .	174
11.1. Recurrence relations	176
11.2. Orthogonality relations for the Legendre polynomials .	178
11.3. Associated Legendre functions	186
12. Applications from electromagnetism	190
13. Spherical harmonics	194
14. The addition theorem for spherical harmonics	197

Problems	199
General References	204
CHAPTER IV. The Laplace and Other Transforms	205
1. Introduction	205
2. Laplace transforms and some general properties	206
3. Solution of linear differential equations with constant coefficients	215
4. Further theorems and their application	220
5. Solution of the equation $\phi(D)x(t) = F(t)$ by means of the convolution theorem	226
6. Application to partial differential equations	230
7. The finite sine transform	232
8. The simply supported rectangular plate	236
9. Free oscillations of a rectangular plate	239
10. Plate subject to combined lateral load and a uniform compression	240
11. The Fourier transform	241
Problems	244
CHAPTER V. Matrices	250
1. Introduction	250
1.1. Definitions	250
2. Determinants	259
2.1. Evaluation of determinants	261
a. Expansion by rows	261
b. Pivotal condensation	263
3. Reciprocal of a square matrix	265
3.1. Determinant of the adjoint matrix	267
4. Solution of simultaneous linear equations	268
4.1. Choleski-Turing method	270
4.2. A special case: the matrix A is symmetric	279
5. Eigenvalues (Latent roots)	280
5.1. The Cayley-Hamilton theorem	281
5.2. Iterative method for determination of eigenvalues .	283
5.3. Evaluation of subdominant eigenvalue	288
6. Special types of matrices	294
6.1. Orthogonal matrix	295
6.2. Hermitian matrix	296
7. Simultaneous diagonalization of two symmetric matrices .	298
Problems	303
General References	308
CHAPTER VI. Analytical Methods in Classical and Wave Mechanics .	309
1. Introduction	309
2. Definitions	309

3. Lagrange's equations of motion for holonomic systems	312
3.1. Derivation of the equations	312
3.2. Conservative forces	315
3.3. Illustrative examples	317
3.4. Energy equation	325
3.5. Orbital motion	326
3.6. The symmetrical top	328
3.7. The two-body problem	330
3.8. Velocity-dependent potentials	335
3.9. The relativistic Lagrangian	335
4. Hamilton's equations of motion	336
5. Motion of a charged particle in an electromagnetic field .	340
6. The solution of the Schrödinger equation	342
6.1. The linear harmonic oscillator	343
6.2. Spherically symmetric potentials in three dimensions .	345
6.3. Two-body problems	351
Problems	356
General References	361
CHAPTER VII. Calculus of Variations	362
1. Introduction	362
2. The fundamental problem: fixed end-points	362
2.1. Special cases	366
2.2. Variable end-points	372
2.3. A generalization of the fixed end-point problem .	375
2.4. One independent, several dependent variables . .	378
2.5. One dependent and several independent variables.	379
3. Isoperimetric problems	382
4. Rayleigh-Ritz method	394
4.1. Sturm-Liouville theory for fourth-order equations .	402
5. Torsion and viscous flow problems	404
5.1. Torsional rigidity	408
5.2. Trefftz method	409
5.3. Generalization to three dimensions	414
6. Variational approach to elastic plate problems	417
6.1. Boundary conditions	419
6.2. Buckling of plates	421
7. Binding energy of the He^4 nucleus	421
8. The approximate solution of differential equations	425
Problems	425
General References	431
CHAPTER VIII. Complex Variable Theory and Conformal Transformations	432
1. The Argand diagram	432
2. Definitions of fundamental operations	435

3. Function of a complex variable	436
3.1. Cauchy-Riemann equations	438
4. Geometry of complex plane	441
5. Complex potential	443
5.1. Uniform stream	444
5.2. Source, sink and vortex	445
5.3. Doublet (Dipole)	452
5.4. Uniform flow + doublet + vortex. Flow past a cylinder	453
5.5. A torsion problem in elasticity	456
6. Conformal transformation	458
6.1. Bilinear (Möbius) transformation	461
7. Schwarz-Christoffel transformation	463
7.1. Applications	467
7.2. The Kirchhoff plane	480
8. Transformation of a circle into an aerofoil	485
Problems	488
General References	495
CHAPTER IX. The Calculus of Residues	496
1. Definition of integration	496
2. Cauchy's theorem	498
3. Cauchy's integral	499
3.1. Differentiation	500
4. Series expansions	501
4.1. Laurent's theorem	503
5. Zeros and singularities	504
5.1. Residues	508
6. Cauchy residue theorem	508
6.1. Application of Cauchy's theorem	510
6.2. Flow round a cylinder	512
6.3. Definite integrals. Integration round unit circle . . .	517
6.4. Infinite integrals	519
6.5. Jordan's lemma	523
6.6. Another type of infinite integral	528
7. Harnack's theorem and applications	531
7.1. The Schwarz and Poisson formulas	534
7.2. Application of conformal transformation to solution of a torsion problem	536
8. Location of zeros of $f(z)$	539
8.1. Nyquist stability criterion	541
9. Summation of series by contour integration	543
10. Representation of functions by contour integrals	550
10.1. Gamma function	551
10.2. Bessel functions	552
10.3. Legendre's function as a contour integral	561
11. Asymptotic expansions	562
12. Saddle-point method	566

Problems	571
General References	577
CHAPTER X. Transform Theory	578
1. Introduction	578
1.1. Complex Fourier transform	578
1.2. Laplace transform	579
1.3. Hilbert transform	580
1.4. Hankel transform	580
1.5. Mellin transform	581
2. Fourier's integral theorem	581
3. Inversion formulas	584
3.1. Complex Fourier transform	584
3.2. Fourier sine and cosine transforms	592
3.3. Convolution theorems for Fourier transforms	593
4. Laplace transform	599
4.1. The inversion integral on the infinite circle	601
4.2. Exercises in the use of the Laplace transform	603
4.3. Linear approximation to axially symmetrical supersonic flow	608
a. Jet in a cylindrical tube	609
b. Quasi-cylindrical free jet	610
4.4. Supersonic flow round a slender body of revolution	612
5. Mixed transforms	615
5.1. Linearized supersonic flow past rectangular symmetrical aerofoil	615
5.2. Heat conduction in a wedge	621
6. Integral equations	626
6.1. The solution of a certain type of integral equation of the first kind	627
6.2. Poisson's integral equation	628
6.3. Abel's integral equation	628
7. Hilbert transforms	633
7.1. Infinite Hilbert transform	633
7.2. Finite Hilbert transform	637
7.3. Alternative forms of the finite Hilbert transform	640
Problems	644
General References	650
CHAPTER XI. Numerical Methods	651
1. Introduction	651
1.1. Finite difference operators	651
2. Interpolation and extrapolation	657
2.1. Linear interpolation	657
2.2. Everett's and Bessel's interpolation formulas	657
2.3. Inverse interpolation	662

2.4. Lagrange interpolation formula	662
2.5. Formulas involving forward or backward differences .	663
3. Some basic expansions	664
4. Numerical differentiation	667
5. Numerical evaluation of integrals	669
5.1. Note on limits of integration	682
5.2. Evaluation of double integrals	682
6. Euler-Maclaurin integration formula	683
6.1. Summation of series	684
7. Solution of ordinary differential equations by means of Taylor series	687
8. Step-by-step method of integration for first-order equations	691
8.1. Simultaneous first-order equations and second-order equations with the first derivative present	694
8.2. The second-order equation $y'' = f(x,y)$	697
8.3. Alternative method for the linear equation $y'' = g(x)y + h(x)$	700
9. Boundary value problems for ordinary differential equations of the second order	701
9.1. Approximate solution of eigenvalue problems by finite differences	702
9.2. Numerical solution of eigenvalue equations	705
10. Linear difference equations with constant coefficients . .	707
11. Finite differences in two dimensions	709
Problems	711
General References	715
CHAPTER XII. Integral Equations	716
1. Introduction	716
1.1. Types of integral equations	716
a. Volterra integral equations	717
b. Fredholm integral equations	717
c. Singular integral equations	718
1.2. Some simple examples of linear integral equations .	718
2. Volterra integral equation form for a differential equation .	722
2.1. Higher order equations	728
3. Fredholm integral equation form for Sturm-Liouville differential equations	731
3.1. The modified Green's function	738
3.2. Green's function for fourth-order differential equations	747
4. Numerical solution	748
4.1. The numerical solution of the homogeneous equation	752
4.2. The Volterra equation	754
4.3. Iteration method of solution	756
a. Fredholm equation	756
b. Volterra equation	757
5. The variation-iteration method for eigenvalue problems .	760

Problems	767
General References	771
Appendix	772
1. $\nabla^2\phi$ in spherical and cylindrical polar coordinates	772
1.1. Plane polar coordinates	772
1.2. Cylindrical polar coordinates	773
1.3. Spherical polar coordinates	773
2. Partial fractions	774
3. Sequences, series, and products	775
3.1. Sequences	775
3.2. Series	776
3.3. Infinite products	788
4. Maxima and minima for functions of two variables	790
4.1. Euler's theorem of homogeneous functions	791
4.2. The expansion of $(\sinh aU)/(\sinh U)$ in powers of $2 \sinh(\frac{1}{2}U)$	792
5. Integration	794
5.1. Uniform convergence of infinite integrals	794
a. Weierstrass M -test	795
b. Dirichlet's test	795
c. Abel's test	795
5.2. Change of variables in a double integral	798
5.3. Special integrals	801
a. Error functions: $\text{erf } x$	801
b. Complementary error function: $\text{erfc } x$	801
c. Gamma function $\Gamma(x)$	801
d. Beta function $B(m,n)$	802
e. The Airy integral	803
5.4. Elliptic integrals	812
6. Principal valued integrals	816
7. Vector algebra and calculus	819
7.1. Curvilinear coordinates	827
7.2. The equation of heat conduction	831
7.3. Components of velocity and acceleration in plane polar coordinates	832
7.4. Vectors, dyads and tensors	835
8. Legendre functions of non-integral order	840
8.1. The value of $P_\nu(0)$	844
9. An equivalent form for $F(a,b;c;x)$	845
10. Integrals involving $L_n^{(k)}(x)$	846
Problems	848
General References	855
Solutions of Problems	856
Chapter I	856
Chapter II	857

CONTENTS**xvii**

Chapter III	860
Chapter IV	861
Chapter V	863
Chapter VI	864
Chapter VII	865
Chapter VIII	866
Chapter IX	867
Chapter X	867
Chapter XI	869
Chapter XII	871
Appendix	872
Subject Index	875