Contents

1. Introduction 1

1.1 The Van der Pol oscillator 2
1.2 Mechanical prototypes of relaxation oscillators 5
1.3 Relaxation oscillations in physics and biology 7
1.4 Discontinuous approximations 9
1.5 Matched asymptotic expansions 12
1.6 Forced oscillations 15
1.7 Mutual entrainment 18

2. Free oscillation 23

2.1 Autonomous relaxation oscillation: definition and existence 25
2.1.1 A mathematical characterization of relaxation oscillations 26
2.1.2 Application of the Poincaré-Bendixson theorem 28
2.1.3 Application of the extension theorem 34
2.1.4 Application of Tikhonov's theorem 42
2.1.5 The analytical method of Cartwright 52
2.2 Asymptotic solution of the Van der Pol equation 55
2.2.1 The physical plane 56
2.2.2 The phase plane 59
2.2.3 The Lienard plane 67
2.2.4 Approximations of amplitude and period 70
2.3 The Volterra-Lotka equations 72
2.3.1 Modeling prey-predator systems 72
2.3.2 Oscillations with both state variables having a large amplitude 74
2.3.3 Oscillations with one state variable having a large amplitude 76
2.3.4 The period for large amplitude oscillations by inverse Laplace asymptotics 83
2.4 Chemical oscillations 87
2.4.1 The Brusselator 87
2.4.2 The Belousov-Zhabotinskii reaction and the Oregonator 89
2.5 Bifurcation of the Van der Pol equation with a constant forcing term 91
2.5.1 Modeling nerve excitation; the Bonhoeffer-Van der Pol equation 92
2.5.2 Canards 94
2.6 Stochastic and chaotic oscillations 99
2.6.1 Chaotic relaxation oscillations 101
2.6.2 Randomly perturbed oscillations 105
2.6.3 The Van der Pol oscillator with a random forcing term 107
2.6.4 Distinction between chaos and noise 112

3. Forced oscillation and mutual entrainment 115

3.1 Modeling coupled oscillations 117
3.1.1 Oscillations in the applied sciences 117
3.1.2 The system of differential equations and the method of analysis 118
3.2 A rigorous theory for weakly coupled oscillators 123
3.2.1 Validity of the discontinuous approximation 123
3.2.2 Construction of the asymptotic solution 125
3.2.3 Existence of a periodic solution 128
3.2.4 Formal extension to oscillators coupled with delay 130
3.3 Coupling of two oscillators 131
3.3.1 Piece-wise linear oscillators 132
3.3.2 Van der Pol oscillators 134
3.3.3 Entrainment with frequency ratio 1:3 136
3.3.4 Oscillators with different limit cycles 138
3.4 Modeling biological oscillations 139
3.4.1 Entrainment with frequency ratio n:m 140
3.4.2 A chain of oscillators with decreasing autonomous frequency 141
3.4.3 A large population of coupled oscillators with widely different frequencies 143
3.4.4 A large population of coupled oscillators with frequencies having a Gaussian distribution 144
3.4.5 Periodic structures of coupled oscillators 144
3.4.6 Nonlinear phase diffusion equations 148

4. The Van der Pol oscillator with a sinusoidal forcing term 151

4.1 Qualitative methods of analysis 153
4.1.1 Global behavior and the Poincaré mapping 154
4.1.2 The use of symbolic dynamics 157
4.1.3 Some remarks on the annulus mapping 158
4.2 Asymptotic solution of the Van der Pol equation with a moderate forcing term 159
4.2.1 Subharmonic solutions 160
4.2.2 Dips slices and chaotic solutions 167
4.3 Asymptotic solution of the Van der Pol equation with a large forcing term 169
4.3.1 Subharmonic solutions 170
4.3.2 Dips and slices 179
4.3.3 Irregular solutions 182

Appendices

A: Asymptotics of some special functions 187
B: Asymptotic ordering and expansions 189
C: Concepts of the theory of dynamical systems 190
D: Stochastic differential equations and diffusion approximations 196

Literature 201

Author Index 215

Subject Index 219