



## Table of Contents

<i>List of abbreviations</i> .....	<i>xi</i>
<i>Foreword</i> .....	<i>xiii</i>
<i>Introduction</i> .....	<i>xvii</i>

### Chapter I - Geometric setting

<b>Introduction</b> .....	<b>1</b>
<b>a): Equations and functions as geometrical objects</b>	
1. The jet space .....	2
2. Contact structure of jet space .....	4
3. Differential equations, solutions, integral manifolds .....	6
4. The group $\text{Diff}(M)$ and its algebra .....	9
5. The groups $\text{Diff}(M^{(m)})$ and $\text{Diff}^{(m)}(M)$ .....	10
6. Group action on algebraic functions and their symmetry .....	11
7. Group action on prolonged functions and the prolongation formula .....	12
<b>b): Symmetry</b>	
8. Symmetries of algebraic equations .....	13
9. Symmetries of differential equations .....	14
10. Symmetry of $\Delta$ versus symmetry of $\Delta = 0$ .....	15
11. Algebras and prolonged algebras .....	16
12. Module structure of symmetry of algebraic equations and first order ODEs .....	17

13. Summary of groups, algebras, and relations .....	19
References .....	21

## **Chapter II - Symmetries and their use**

Introduction .....	23
1. Symmetry of a given equation .....	24
2. Linear and C-linearizable equations .....	26
3. Equations with a given symmetry .....	28
4. Canonical coordinates .....	31
5. Symmetry and reduction of algebraic equations .....	33
6. Symmetry and reduction of ODEs .....	36
7. Symmetry and symmetric solutions of PDEs .....	38
8. Conditional symmetries .....	40
9. Conditional symmetries and boundary conditions .....	40
References .....	43

## **Chapter III - Examples**

Introduction .....	45
1. Symmetry of algebraic equations .....	45
2. Symmetry of ODEs (one-soliton KdV) .....	47
3. Symmetry of evolution PDEs (the heat equation) .....	49
4. Table of prolongations for ODEs .....	52
5. Table of prolongations for PDEs .....	53

## **Chapter IV - Evolution equations**

Introduction .....	55
a): Evolution equations - general features	
1. Evolution equations .....	56

2. Special classes of symmetries .....	58
3. Contact transformations .....	59
4. Autonomous equations .....	60
<i>b): Dynamical systems (ODEs)</i>	
5. First order ODEs .....	61
6. Autonomous equations, tangent bundle versus jet space, topology of solutions, and time-independent symmetries .....	62
7. Equations in Lax form .....	64
8. Second order ODEs .....	66
9. Lagrange versus Hamilton equations .....	68
10. Potential systems .....	70
11. Higher order ODEs .....	71
<i>c): Periodic solutions</i>	
12. Periodic solutions of autonomous dynamical systems .....	72
13. Periodic solutions of potential systems .....	74
14. Point particles on the circle .....	75
<i>d): Evolution PDEs</i>	
15. First order PDEs .....	76
16. Higher order evolution equations .....	78
17. Scalar equations linear in higher derivatives .....	79
18. Equations linear in higher derivatives .....	80
References .....	81
<b>Chapter V - Variational problems</b>	
Introduction .....	83
1. Variational symmetries and variational problems .....	84

<b>2. Variational symmetries and conservation laws:</b>	
Lagrangian mechanics and Noether theorem .....	86
<b>3. Conserved quantities for higher order variational problems: the general Noether theorem</b> .....	88
<b>4. Noether theorem and divergence symmetries</b> .....	90
<b>5. Variational symmetries and reduction of order</b> .....	91
<b>6. Variational symmetries, conservation laws, and the Noether theorem for infinite dimensional variational problems</b> .....	92
<b>References</b> .....	95

### **Chapter VI - Bifurcation problems**

<b>Introduction</b> .....	97
<b>1. Bifurcation problems: general setting</b> .....	98
<b>2. Bifurcation theory and linear symmetry</b> .....	99
<b>3. Lie-point symmetries and bifurcation</b> .....	104
<b>4. Symmetries of systems of ODEs depending on a parameter</b> .....	112
<b>5. Bifurcation points and symmetry algebra</b> .....	117
<b>6. Extensions</b> .....	119
<b>References</b> .....	120

### **Chapter VII - Gauge theories**

<b>Introduction</b> .....	123
<b>1. Symmetry breaking in potential problems and gauge theories</b> .....	124
<b>2. Strata in <math>R^N</math></b> .....	126
<b>3. Michel's theorem</b> .....	127
<b>4. Zero-th order gauge functionals</b> .....	129
<b>5. Discussion</b> .....	131
<b>6. First order gauge functionals</b> .....	132
<b>7. Geometry and stratification of <math>\Omega</math></b> .....	137
<b>8. Stratification of gauge orbit space</b> .....	139
<b>9. Maximal strata in gauge orbit space</b> .....	142

10. The equivariant branching lemma .....	144
11. A reduction lemma for gauge invariant potentials .....	146
12. Some examples of reduction .....	148
13. Base space symmetries .....	149
14. A scenario for pattern formation .....	151
15. A scenario for phase coexistence .....	152
References .....	153

## **Chapter VIII - Reduction and equivariant branching lemma**

Introduction .....	155
1. General setting (ODEs) .....	156
2. The reduction lemma .....	157
3. The equivariant branching lemma .....	158
4. General setting (PDEs) .....	160
5. Gauge symmetries and Lie point vector fields .....	161
6. Reduction lemma for gauge theories .....	162
7. Symmetric critical sections of gauge functionals .....	165
8. Equivariant branching lemma for gauge functionals .....	165
9. Evolution PDEs .....	167
10. Symmetries of evolution PDEs .....	168
11. Reduction lemma for evolution PDEs .....	171
References .....	172

## **Chapter IX - Further developments**

Introduction .....	175
1. Missing sections .....	176
2. Non Linear Superposition Principles .....	177
3. Symmetry and integrability - second order ODEs .....	180
4. Infinite dimensional (and Kac-Moody) Lie-point symmetry algebras .....	180

<b>5. Symmetry classification of ODEs .....</b>	<b>183</b>
<b>6. The Lie determinant .....</b>	<b>185</b>
<b>7. Systems of linear second order ODEs .....</b>	<b>187</b>
<b>8. Cohomology and symmetry of differential equations .....</b>	<b>189</b>
<b>9. Contact symmetries of evolution equations .....</b>	<b>192</b>
<b>10. Conditional symmetries, and Boussinesq equation .....</b>	<b>194</b>
<b>11. Lie point symmetries and maps .....</b>	<b>197</b>
<b>References .....</b>	<b>200</b>

## **Chapter X - Equations of Physics**

<b>Introduction .....</b>	<b>205</b>
<b>1. Fokker-Planck type equations .....</b>	<b>206</b>
<b>2. Schroedinger equation for atoms and molecules .....</b>	<b>208</b>
<b>3. Einstein (vacuum) field equations .....</b>	<b>209</b>
<b>4. Landau-Ginsburg equation .....</b>	<b>210</b>
<b>5. The <math>\Phi^6</math> field theory (three dimensional Landau-Ginzburg equation) .....</b>	<b>212</b>
<b>6. An equation arising in plasma physics .....</b>	<b>214</b>
<b>7. Navier-Stokes equations .....</b>	<b>215</b>
<b>8. Yang-Mills equations .....</b>	<b>216</b>
<b>9. Lattice equations and the Toda lattice .....</b>	<b>218</b>
<b>References .....</b>	<b>220</b>
<b>References and bibliography .....</b>	<b>223</b>
<b>Subject Index .....</b>	<b>253</b>

