Contents

Symplectic Geometry
V.I. Arnol'd, A.B. Givental'
1

Geometric Quantization
A.A. Kirillov
137

Integrable Systems. I
B.A. Dubrovin, I.M. Krichever, S.P. Novikov
173

Index
281

Symplectic Geometry

V. I. Arnol'd, A. B. Givental'

Translated from the Russian
by G. Wassermann

Contents

Foreword 4

Chapter 1. Linear Symplectic Geometry 5

§1. Symplectic Space 5
1.1. The Skew-Scalar Product 5
1.2. Subspaces 5
1.3. The Lagrangian Grassmann Manifold 6
§2. Linear Hamiltonian Systems 7
2.1. The Symplectic Group and its Lie Algebra 7
2.2. The Complex Classification of Hamiltonians 8
2.3. Linear Variational Problems 9
2.4. Normal Forms of Real Quadratic Hamiltonians 10
2.5. Sign-Definite Hamiltonians and the Minimax Principle 11
§3. Families of Quadratic Hamiltonians 12
3.1. The Concept of the Miniversal Deformation 12
3.2. Miniversal Deformations of Quadratic Hamiltonians 13
3.3. Generic Families 14
3.4. Bifurcation Diagrams 16
§4. The Symplectic Group 17
4.1. The Spectrum of a Symplectic Transformation 17
4.2. The Exponential Mapping and the Cayley Parametrization 18
4.3. Subgroups of the Symplectic Group 18
4.4. The Topology of the Symplectic Group 19
4.5. Linear Hamiltonian Systems with Periodic Coefficients 19

Chapter 2. Symplectic Manifolds 22

§1. Local Symplectic Geometry 22
1.1. The Darboux Theorem 22
1.2. Example. The Degeneracies of Closed 2-Forms on $\mathbb{R}^4$ 23
1.3. Germs of Submanifolds of Symplectic Space 24
1.4. The Classification of Submanifold Germs 25
1.5. The Exterior Geometry of Submanifolds 26
1.6. The Complex Case 27
§2. Examples of Symplectic Manifolds 27
2.1. Cotangent Bundles 27
2.2. Complex Projective Manifolds 28
2.3. Symplectic and Kähler Manifolds 29
2.4. The Orbits of the Coadjoint Action of a Lie Group 30
§3. The Poisson Bracket 31
3.1. The Lie Algebra of Hamiltonian Functions 31
3.2. Poisson Manifolds 32
3.3. Linear Poisson Structures 33
3.4. The Linearization Problem 34
§4. Lagrangian Submanifolds and Fibrations 35
4.1. Examples of Lagrangian Manifolds 35
4.2. Lagrangian Fibrations 36
4.3. Intersections of Lagrangian Manifolds and Fixed Points of Symplectomorphisms 38

Chapter 3. Symplectic Geometry and Mechanics 42

§1. Variational Principles 42
1.1. Lagrangian Mechanics 43
1.2. Hamiltonian Mechanics 44
1.3. The Principle of Least Action 45
1.4. Variational Problems with Higher Derivatives 46
1.5. The Manifold of Characteristics 48
1.6. The Shortest Way Around an Obstacle 49
§2. Completely Integrable Systems 51
2.1. Integrability According to Liouville 51
2.2. The "Action-Angle" Variables 53
2.3. Elliptical Coordinates and Geodesics on an Ellipsoid 54
2.4. Poisson Pairs 57
2.5. Functions in Involution on the Orbits of a Lie Coalgebra 58
2.6. The Lax Representation 59
§3. Hamiltonian Systems with Symmetries 61
3.1. Poisson Actions and Momentum Mappings 61
3.2. The Reduced Phase Space and Reduced Hamiltonians 62
3.3. Hidden Symmetries 63

3.4. Poisson Groups 65
3.5. Geodesics of Left-Invariant Metrics and the Euler Equation ... 66
3.6. Relative Equilibria ... 66
3.7. Noncommutative Integrability of Hamiltonian Systems ... 67
3.8. Poisson Actions of Tori ... 68

Chapter 4. Contact Geometry ... 71

§1. Contact Manifolds ... 71
1.1. Contact Structure ... 71
1.2. Examples 72
1.3. The Geometry of the Submanifolds of a Contact Space ... 74
1.4. Degeneracies of Differential 1-Forms on $\mathbb{R}^n$ ... 76
§2. Symplectification and Contact Hamiltonians ... 77
2.1. Symplectification ... 77
2.2. The Lie Algebra of Infinitesimal Contactomorphisms ... 79
2.3. Contactification ... 80
2.4. Lagrangian Embeddings in $\mathbb{R}^{2n}$ ... 81
§3. The Method of Characteristics ... 82
3.1. Characteristics on a Hypersurface in a Contact Space ... 82
3.2. The First-Order Partial Differential Equation ... 83
3.3. Geometrical Optics ... 84
3.4. The Hamilton–Jacobi Equation ... 85

Chapter 5. Lagrangian and Legendre Singularities ... 87

§1. Lagrangian and Legendre Mappings ... 87
1.1 Fronts and Legendre Mappings ... 87
1.2. Generating Families of Hypersurfaces ... 89
1.3. Caustics and Lagrangian Mappings ... 91
1.4. Generating Families of Functions ... 92
1.5. Summary ... 93
§2. The Classification of Critical Points of Functions ... 94
2.1. Versal Deformations: An Informal Description ... 94
2.2. Critical Points of Functions ... 95
2.3. Simple Singularities ... 97
2.4. The Platonics ... 98
2.5. Miniversal Deformations ... 98
§3. Singularities of Wave Fronts and Caustics ... 99
3.1 The Classification of Singularities of Wave Fronts and Caustics in Small Dimensions ... 99
3.2. Boundary Singularities ... 101
3.3. Weyl Groups and Simple Fronts ... 104
3.4. Metamorphoses of Wave Fronts and Caustics ... 106
3.5. Fronts in the Problem of Going Around an Obstacle ... 109

Chapter 6. Lagrangian and Legendre Cobordisms 113
§ 1. The Maslov Index 113
1.1. The Quasiclassical Asymptotics of the Solutions of the Schrödinger Equation 114
1.2. The Morse Index and the Maslov Index 115
1.3. The Maslov Index of Closed Curves 116
1.4. The Lagrangian Grassmann Manifold and the Universal Maslov Class 117
1.5. Cobordisms of Wave Fronts on the Plane 119
§ 2. Cobordisms 121
2.1. The Lagrangian and the Legendre Boundary 121
2.2. The Ring of Cobordism Classes 122
2.3. Vector Bundles with a Trivial Complexification 122
2.4. Cobordisms of Smooth Manifolds 123
2.5. The Legendre Cobordism Groups as Homotopy Groups 124
2.6. The Lagrangian Cobordism Groups 125
§ 3. Characteristic Numbers 126
3.1. Characteristic Classes of Vector Bundles 126
3.2. The Characteristic Numbers of Cobordism Classes 127
3.3. Complexes of Singularities 128
3.4. Coexistence of Singularities 129
References . . . 131

Geometric Quantization

A. A. Kirillov

Translated from the Russian
by G. Wassermann

Contents

Introduction 138

§1. Statement of the Problem. 138
1.1. The Mathematical Model of Classical Mechanics in the Hamiltonian Formalism 138
1.2. The Mathematical Model of Quantum Mechanics 143
1.3. The Statement of the Quantization Problem. The Connection with the Method of Orbits in Representation Theory 145
§2. Prequantization 146
2.1. The Koopman–Van Hove–Segal Representation. 146
2.2. Hermitian Bundles with a Connection. The Souriau–Kostant Prequantization 147
2.3. Examples. Prequantization of the Two-Dimensional Sphere and the Two-Dimensional Torus. 151
2.4. Prequantization of Symplectic Supermanifolds 153
§3. Polarizations . . . 153
3.1. The Definition of a Polarization 153
3.2. Polarizations on Homogeneous Manifolds 155
§4. Quantization 157
4.1. The Space of a Quantization. 157
4.2. Quantization of a Flat Space 159
4.3. The Connection with the Maslov Index and with the Weil Representation 165

4.4. The General Scheme of Geometric Quantization 167
4.5. The Quantization Operators 168
References 170

Integrable Systems. I

B.A. Dubrovin, I.M. Krichever, S.P. Novikov

Translated from the Russian
by G. Wassermann

Contents

Introduction 174

Chapter 1. Hamiltonian Systems. Classical Methods of Integration . . 175

§ 1. The General Concept of the Poisson Bracket. The Principal Examples 175
§ 2. Integrals and Reduction of the Order of Hamiltonian Systems. Systems with Symmetry 190
§ 3. Liouville's Theorem. Action-Angle Variables 202
§ 4. The Hamilton–Jacobi Equation. The Method of Separation of Variables—The Classical Method of Integration and of Finding Action-Angle Variables. 205

Chapter 2. Modern Ideas on the Integrability of Evolution Systems . . 208

§ 1. Commutational Representations of Evolution Systems 208
§ 2. Algebraic-Geometric Integrability of Finite-Dimensional λ-Families 222
§ 3. The Hamiltonian Theory of Hyperelliptic λ-Families. 238
§ 4. The Most Important Examples of Systems Integrable by Two-Dimensional Theta Functions 245
§ 5. Pole Systems 256
§ 6. Integrable Systems and the Algebraic-Geometric Spectral Theory of Linear Periodic Operators 260

References. 271