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I. Ordinary Differential Equations

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II. Smooth Dynamical Systems

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Translated from the Russian
by E. R. Dawson and D. O'Shea

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