CONTENTS

I. Review of Fundamental Notions of Analysis 1

A. Set Theory, Definitions 1
1. Sets 1
2. Mappings 2
3. Relations 5
4. Orderings 5

B. Algebraic Structures, Definitions 6
1. Groups 7
2. Rings 7
3. Modules 8
4. Algebras 9
5. Linear spaces 9

C. Topology 11
1. Definitions 11
2. Separation 13
3. Base 13
4. Convergence 14
5. Covering and compactness 15
6. Connectedness 16
7. Continuous mappings 16
8. Multiple connectedness 18
9. Associated topologies 20
10. Topology related to other structures 20
11. Metric spaces 22
metric spaces 22
Cauchy sequence; completeness 24
12. Banach spaces 25
normed linear spaces 25
Banach spaces 28
strong and weak topology; compactedness 29
13. Hilbert spaces 30

D. Integration 30
1. Introduction 31
2. Measures 32
3. Measure spaces 34

4. Measurable functions 39
5. Integrable functions 40
6. Integration on locally compact spaces 45
7. Signed and complex measures 48
8. Integration of vector valued functions 49
9. L^1 space 51
10. L^p space 52

E. Key Theorems in Linear Functional Analysis 56
1. Bounded linear operators 56
2. Compact operators 60
3. Open mapping and closed graph theorems 62

Problems and Exercises 63
Problem 1: Clifford algebra, Spin(4) 63
Exercise 2: Product topology 67
Problem 3: Strong and weak topologies in L^2 68
Exercise 4: Hölder spaces 69
See Problem VI 4: Application to the Schrödinger equation

II. Differential Calculus on Banach Spaces 71

A. Foundations 71
1. Definitions. Taylor expansion 71
2. Theorems 73
3. Diffeomorphisms 74
4. The Euler equation 76
5. The mean value theorem 78
6. Higher order differentials 79

B. Calculus of Variations 82
1. Necessary conditions for minima 82
2. Sufficient conditions 83
3. Lagrangian problems 86

C. Implicit Function Theorem. Inverse Function Theorem 88
1. Contracting mapping theorems 88
2. Inverse function theorem 90
3. Implicit function theorem 91
4. Global theorems 92

D. Differential Equations 94
1. First order differential equation 94
2. Existence and uniqueness theorems for the lipschitzian case 95

Problems and Exercises 98
Problem 1: Banach spaces, first variation, linearized equation 98
Problem 2: Taylor expansion of the action; Jacobi fields; the Feynman-Green function; the Van Vleck matrix; conjugate points; caustics 100
Problem 3: Euler-Lagrange equation; the small disturbance equation; the soap bubble problem; Jacobi fields 105

III. Differentiable Manifolds, Finite Dimensional Case 111

A. Definitions 111
1. Differentiable manifolds 111
2. Diffeomorphisms 115
3. Lie groups 116

B. Vector Fields; Tensor Fields 117
1. Tangent vector space at a point 117
tangent vector as a derivation 117
tangent vector defined by transformation properties 119
tangent vector as an equivalence class of curves 120
images under differentiable mappings 121
2. Fibre bundles 124
definition 124
bundle morphisms 126
tangent bundle 127
frame bundle 128
principal fibre bundle 128
3. Vector fields 130
vector fields 130
moving frames 132
images under diffeomorphisms 132
4. Covariant vectors; cotangent bundles 133
dual of the tangent space 133
space of differentials 135
cotangent bundle 135
reciprocal images 136
5. Tensors at a point 136
tensors at a point 136
tensor algebra 138
6. Tensor bundles; tensor fields 140

C. Groups of Transformations 141
1. Vector fields as generators of transformation groups 141
2. Lie derivatives 145
3. Invariant tensor fields 148

D. Lie Groups 150
1. Definitions; notations 150
2. Left and right translations; Lie algebra; structure constants 153
3. One-parameter subgroups 155
4. Exponential mapping; Taylor expansion; canonical coordinates 157
5. Lie groups of transformations; realization 159
6. Adjoint representation 161
7. Canonical form 162

Problems and Exercises 162
Problem 1: Change of coordinates on a fiber bundle, configuration space, phase space 162
Problem 2: Lie algebras of Lie groups 165
Problem 3: The strain tensor 170
Problem 4: The group manifolds of SO(3) and SU(2) 171
Problem 5: The 2-sphere 181

IV. Integration on Manifolds 187

A. Exterior Differential Forms 187
1. Exterior algebra 187
exterior product 188
local coordinates; strict components 189
change of basis 190
2. Exterior differentiation 191
3. Reciprocal image of a form (pull back) 195
4. Derivations and antiderivations 197
definitions 197
interior product 198
5. Forms defined on a Lie group 199
invariant forms 199
Maurer–Cartan structure equations 200

B. Integration 201
1. Integration 201
orientation 201
odd forms 202
integration of n-forms in $\mathbb{R}^n$ 202
partitions of unity 203
properties of integrals 204
2. Stokes' theorem 205
p-chains 206
integrals of p-forms on p-chains 207
boundaries 207
mappings of chains 209

proof of Stokes' theorem 211
3. Global properties 212
homology and cohomology 212
0-forms and 0-chains 213
Betti numbers 213
Poincaré lemmas 213
de Rham and Poincaré duality theorems 216

C. Exterior Differential Systems 218
1. Exterior equations 218
2. Single exterior equation 218
3. Systems of exterior equations 221
ideal generated by a system of exterior equations 221
algebraic equivalence 221
solutions 222
examples 224
4. Exterior differential equations 225
integral manifolds 225
associated Pfaff systems 226
generic points 226
closure 226
5. Mappings of manifolds 227
introduction 227
immersion 229
embedding 229
submersion 230
6. Pfaff systems 230
complete integrability 231
Frobenius theorem 232
integrability criterion 233
examples 234
dual form of the Frobenius theorem 236
7. Characteristic system 238
characteristic manifold 238
example: first order partial differential equations 238
complete integrability 241
construction of integral manifolds 242
Cauchy problem 244
examples 247
8. Invariants 249
invariant with respect to a Pfaff system 249
integral invariants 251
9. Example: Integral invariants of classical dynamics 253
Liouville theorem 254
canonical transformations 255
10. Symplectic structures and hamiltonian systems 255

Problems and Exercises 258
Problem 1: Compound matrices 258
Problem 2: Poincaré lemma, Maxwell equations, wormholes 259
Problem 3: Integral manifolds 259
Problem 4: First order partial differential equations, Hamilton–Jacobi equations, lagrangian manifolds 260
Problem 5: First order partial differential equations, catastrophes 265
Problem 6: Darboux theorem 269
Problem 7: Time dependent hamiltonians 271
See Problem VI 11 paragraph c: Electromagnetic shock waves

V. Riemannian Manifolds 273

A. The Riemannian Structure 273
1. Preliminaries 273
metric tensor 273
hyperbolic manifold 275
induced metric 278
2. Existence of a riemannian structure 280
proper structure 280
hyperbolic structure 280
Euler–Poincaré characteristic 281
3. Star operator 281
volume element 282
star operator 283
4. Isometries 285

B. Connections 287
1. Connections on a principle fibre bundle 287
definitions 287
covariant derivative, local coordinates 289
existence of a connection 290
curvature 291
Cartan structure equations 291
Bianchi identities 292
definitions: canonical form and torsion form 293
2. Linear connections 293
covariant derivative 294
connection forms 294
parallel translation 295
affine geodesic 295
torsion and curvature 298
3. Riemannian connection 300
definitions 301
locally flat manifolds 303

4. Differential operators 305
exterior derivative 305
operator δ 305
divergence 306
laplacian 307

C. Geodesics 308
1. Arc length 308
2. Variations 309
Euler equations 311
energy integral 312
3. Exponential mapping 313
definition 313
normal coordinates 313
4. Geodesics on a proper riemannian manifold 314
properties 314
geodesic completeness 317
5. Geodesics on a hyperbolic manifold 317

Problems and Exercises 318
Problem 1: Maxwell equation; gravitational radiation 318
Problem 2: The Schwarzschild solution 323
Problem 3: Geodetic motion; equation of geodetic deviation; exponentiation; conjugate points 326
Problem 4: Spinors; spin connection 331
Problem 5: Causal structures; conformal spaces; Weyl tensor 336

VI. Distributions 343

A. Test Functions 343
1. Seminorms 343
definitions 343
Hahn–Banach theorem 344
topology defined by a family of seminorms 344
2. $\mathscr{D}$-spaces 347
definitions 347
inductive limit topology 349
convergence in $\mathscr{D}^m(U)$ and $\mathscr{D}(U)$ 350
examples of functions in $\mathscr{D}$ 351
truncating sequences 354
density theorem 354

B. Distributions 355
1. Definitions 355
distributions 355

measures; Dirac measures and Leray forms 358
distribution of order p 359
support of a distribution 361
distributions with compact support 361
2. Operations on distributions 364
sum 364
product by C^∞ function 364
direct product 365
derivations 366
examples 367
inverse derivative 370
3. Topology on $\mathscr{D}'$ 373
weak star topology 373
criterion of convergence 374
4. Change of variables in $\mathbb{R}^n$ 376
change of variables in $\mathbb{R}^n$ 376
transformation of a distribution under a diffeomorphism 377
invariance 379
5. Convolution 379
convolution algebra $L^1(\mathbb{R}^n)$ 379
convolution algebra $\mathscr{D}'^+$ and $\mathscr{D}'^-$ 382
derivation and translation of a convolution product 384
regularization 385
support of a convolution 385
equations of convolution 386
differential equation with constant coefficients 389
systems of convolution equations 390
kernels 391
6. Fourier transform 394
Fourier transform of integrable functions 394
tempered distributions 396
Fourier transform of tempered distributions 396
Paley–Wiener theorem 397
Fourier transform of a convolution 398
7. Distribution on a C^∞ paracompact manifold 400
8. Tensor distributions 402

C. Sobolev Spaces and Partial Differential Equations 406
1. Sobolev spaces 406
properties 407
density theorems 408
W_p^{-m} spaces 409
Fourier transform 410
Plancherel theorem 410
Sobolev's inequalities 411
2. Partial differential equations 412

definitions 412
Cauchy–Kovalevski theorem 413
classifications 414
3. Elliptic equations; laplacians 415
elementary solution of Laplace's equation 415
subharmonic distributions 416
potentials 416
energy integral, Green's formula, unicity theorem 419
Liouville's theorem 420
boundary-value problems 422
Green function 423
introduction to hilbertian methods; generalized Dirichlet problem 425
hilbertian methods 427
example: Neumann problem 429
4. Parabolic equations 430
heat diffusion 430
5. Hyperbolic equation; wave equations 431
elementary solution of the wave equation 431
Cauchy problem 432
energy integral, unicity theorem 433
existence theorem 435
6. Leray theory of hyperbolic systems 436
7. Second order systems; propagators 442

Problems and Exercises 445
Problem 1: Bounded distributions 445
Problem 2: Laplacian of a discontinuous function 447
Exercise 3: Regularized functions 448
Problem 4: Application to the Schrödinger equation 448
Exercise 5: Convolution and linear continuous responses 450
Problem 6: Fourier transforms of $\exp(-x^2)$ and $\exp(ix^2)$ 451
Problem 7: Fourier transforms of Heaviside functions and $\mathrm{Pv}(1/x)$ 452
Problem 8: Dirac bitensors 453
Problem 9: Legendre condition 453
Problem 10: Hyperbolic equations; characteristics 454
Problem 11: Electromagnetic shock waves 455
Problem 12: Elementary solution of the wave equation 458
Problem 13: Elementary kernels of the harmonic oscillator 458

VII. Differentiable Manifolds, Infinite Dimensional Case 463

A. Infinite-Dimensional Manifolds 463
1. Definitions and general properties 463
E-manifolds 463

differentiable functions 464
tangent vector 464
vector and tensor field 465
differential of a mapping 466
submanifold 467
immersion, embedding, submersion 469
flow a vector field 471
differential forms 471
2. Symplectic structures and hamiltonian systems 472
definitions 472
complex structures 472
canonical symplectic form 474
symplectic transformation 474
hamiltonian vector field 474
conservation of energy theorem 475
riemannian manifolds 475

B. Theory of Degree; Leray–Schauder Theory 476
1. Definition for finite dimensional manifolds 477
degree 477
integral formula for the degree of a function 478
continuous mappings 480
2. Properties and applications 481
fundamental theorem 481
Borsuk's theorem 482
Brouwer's fixed point theorem 482
product theorem 483
3. Leray–Schauder theory 483
definitions 483
compact mappings 484
degree of a compact mapping 484
Schauder fixed point theorem 485
Leray–Schauder theorem 485

C. Morse Theory 487
1. Introduction 487
2. Definitions and theorems 487
3. Index of a critical point 491
4. Critical neck theorem 492

D. Cylindrical Measures, Wiener Integral 493
1. Introduction 493
2. Promeasures and measures on a locally convex space 495
projective system 495
promeasures 496
image of a promeasure 498

integration with respect to a promeasure of a cylindrical function 498
Fourier transforms 499
3. Gaussian promeasures 501
gaussian measures on $\mathbb{R}^n$ 501
gaussian promeasures 502
gaussian promeasures on Hilbert spaces 503
4. The Wiener measure 503
Wiener integral 506
sequential Wiener integral 507

Problems and Exercises 509
Problem A: The Klein–Gordon equation 509
Problem B: Application of the Leray–Schauder theorem 511
Problem C: The Reeb theorem 512
Problem D1: A metric on the space of paths with fixed end points 513
Problem D2: Measures invariant under translation 514
Problem D3: Cylindrical σ-field of $C([a, b])$ 514
Problem D4: Generalized Wiener integral of a cylindrical function 515

References 521

Symbols 527

Index 533