



# Contents

## 0 Preliminaries

0.1	Fundamentals on Lebesgue Integration . . . . .	1
0.2	Convolutions on the Line Group . . . . .	4
0.3	Further Sets of Functions and Sequences . . . . .	6
0.4	Periodic Functions and Their Convolution . . . . .	8
0.5	Functions of Bounded Variation on the Line Group . . . . .	10
0.6	The Class $BV_{2\pi}$ . . . . .	14
0.7	Normed Linear Spaces, Bounded Linear Operators . . . . .	15
0.8	Bounded Linear Functionals, Riesz Representation Theorems . . . . .	20
0.9	References	24

## Part I

### Approximation by Singular Integrals . . . . .

## 1 Singular Integrals of Periodic Functions

1.0	Introduction . . . . .	29
1.1	Norm-Convergence and -Derivatives . . . . .	30
1.1.1	Norm-Convergence 30	
1.1.2	Derivatives 33	
1.2	Summation of Fourier Series . . . . .	39
1.2.1	Definitions 39	
1.2.2	Dirichlet and Fejér Kernel 42	
1.2.3	Weierstrass Approximation Theorem 44	
1.2.4	Summability of Fourier Series 44	
1.2.5	Row-Finite $\theta$ -Factors 47	
1.2.6	Summability of Conjugate Series 47	
1.2.7	Fourier-Stieltjes Series 49	
1.3	Test Sets for Norm-Convergence . . . . .	54
1.3.1	Norms of Some Convolution Operators 54	
1.3.2	Some Applications of the Theorem of Banach–Steinhaus 55	
1.3.3	Positive Kernels 58	
1.4	Pointwise Convergence . . . . .	61
1.5	Order of Approximation for Positive Singular Integrals . . . . .	67
1.5.1	Modulus of Continuity and Lipschitz Classes 67	
1.5.2	Direct Approximation Theorems 68	
1.5.3	Method of Test Functions 70	
1.5.4	Asymptotic Properties 72	

<b>1.6</b>	<b>Further Direct Approximation Theorems, Nikolskii Constants . . . . .</b>	<b>79</b>
1.6.1	Singular Integral of Fejér–Korovkin 79	
1.6.2	Further Direct Approximation Theorems 80	
1.6.3	Nikolskii Constants 82	
<b>1.7</b>	<b>Simple Inverse Approximation Theorems . . . . .</b>	<b>86</b>
<b>1.8</b>	<b>Notes and Remarks . . . . .</b>	<b>89</b>
<b>2</b>	<b>Theorems of Jackson and Bernstein for Polynomials of Best Approximation and for Singular Integrals</b>	
<b>2.0</b>	<b>Introduction . . . . .</b>	<b>94</b>
<b>2.1</b>	<b>Polynomials of Best Approximation . . . . .</b>	<b>95</b>
<b>2.2</b>	<b>Theorems of Jackson . . . . .</b>	<b>97</b>
<b>2.3</b>	<b>Theorems of Bernstein . . . . .</b>	<b>99</b>
<b>2.4</b>	<b>Various Applications . . . . .</b>	<b>104</b>
<b>2.5</b>	<b>Approximation Theorems for Singular Integrals . . . . .</b>	<b>109</b>
2.5.1	Singular Integral of Abel–Poisson 109	
2.5.2	Singular Integral of de La Vallée Poussin 112	
<b>2.6</b>	<b>Notes and Remarks . . . . .</b>	<b>116</b>
<b>3</b>	<b>Singular Integrals on the Line Group</b>	
<b>3.0</b>	<b>Introduction . . . . .</b>	<b>119</b>
<b>3.1</b>	<b>Norm-Convergence . . . . .</b>	<b>120</b>
3.1.1	Definitions and Fundamental Properties 120	
3.1.2	Singular Integral of Fejér 122	
3.1.3	Singular Integral of Gauss–Weierstrass 125	
3.1.4	Singular Integral of Cauchy–Poisson 126	
<b>3.2</b>	<b>Pointwise Convergence . . . . .</b>	<b>132</b>
<b>3.3</b>	<b>Order of Approximation . . . . .</b>	<b>136</b>
<b>3.4</b>	<b>Further Direct Approximation Theorems . . . . .</b>	<b>142</b>
<b>3.5</b>	<b>Inverse Approximation Theorems . . . . .</b>	<b>146</b>
<b>3.6</b>	<b>Shape Preserving Properties . . . . .</b>	<b>150</b>
3.6.1	Singular Integral of Gauss–Weierstrass 150	
3.6.2	Variation Diminishing Kernels 154	
<b>3.7</b>	<b>Notes and Remarks . . . . .</b>	<b>158</b>
<b>Part II</b>		
<b>Fourier Transforms . . . . .</b>	<b>163</b>	
<b>4</b>	<b>Finite Fourier Transforms</b>	
<b>4.0</b>	<b>Introduction . . . . .</b>	<b>167</b>
<b>4.1</b>	<b><math>L_{2\pi}^1</math>-Theory . . . . .</b>	<b>167</b>
4.1.1	Fundamental Properties 167	
4.1.2	Inversion Theory 171	
4.1.3	Fourier Transforms of Derivatives 172	
<b>4.2</b>	<b><math>L_{2\pi}^p</math>-Theory, <math>p &gt; 1</math> . . . . .</b>	<b>174</b>
4.2.1	The Case $p = 2$ 174	
4.2.2	The Case $p \neq 2$ 177	

<b>4.3</b>	<b>Finite Fourier-Stieltjes Transforms . . . . .</b>	<b>179</b>
4.3.1	Fundamental Properties 179	
4.3.2	Inversion Theory 182	
4.3.3	Fourier-Stieltjes Transforms of Derivatives 183	
<b>4.4</b>	<b>Notes and Remarks . . . . .</b>	<b>185</b>
<b>5</b>	<b>Fourier Transforms Associated with the Line Group</b>	
<b>5.0</b>	<b>Introduction . . . . .</b>	<b>188</b>
<b>5.1</b>	<b><math>L^1</math>-Theory . . . . .</b>	<b>188</b>
5.1.1	Fundamental Properties 188	
5.1.2	Inversion Theory 190	
5.1.3	Fourier Transforms of Derivatives 194	
5.1.4	Derivatives of Fourier Transforms, Moments of Positive Functions, Peano and Riemann Derivatives 196	
5.1.5	Poisson Summation Formula 201	
<b>5.2</b>	<b><math>L^p</math>-Theory, <math>1 &lt; p \leq 2</math> . . . . .</b>	<b>208</b>
5.2.1	The Case $p = 2$ 208	
5.2.2	The Case $1 < p < 2$ 209	
5.2.3	Fundamental Properties 212	
5.2.4	Summation of the Fourier Inversion Integral 213	
5.2.5	Fourier Transforms of Derivatives 214	
5.2.6	Theorem of Plancherel 216	
<b>5.3</b>	<b>Fourier-Stieltjes Transforms . . . . .</b>	<b>219</b>
5.3.1	Fundamental Properties 219	
5.3.2	Inversion Theory 222	
5.3.3	Fourier-Stieltjes Transforms of Derivatives 224	
<b>5.4</b>	<b>Notes and Remarks . . . . .</b>	<b>227</b>
<b>6</b>	<b>Representation Theorems</b>	
<b>6.0</b>	<b>Introduction . . . . .</b>	<b>231</b>
<b>6.1</b>	<b>Necessary and Sufficient Conditions . . . . .</b>	<b>232</b>
6.1.1	Representation of Sequences as Finite Fourier or Fourier-Stieltjes Transforms 232	
6.1.2	Representation of Functions as Fourier or Fourier-Stieltjes Transforms 235	
<b>6.2</b>	<b>Theorems of Bochner . . . . .</b>	<b>241</b>
<b>6.3</b>	<b>Sufficient Conditions . . . . .</b>	<b>246</b>
6.3.1	Quasi-Convexity 246	
6.3.2	Representation as $L_{2\pi}^1$ -Transform 249	
6.3.3	Representation as $L^1$ -Transform 250	
6.3.4	A Reduction Theorem 252	
<b>6.4</b>	<b>Applications to Singular Integrals . . . . .</b>	<b>256</b>
6.4.1	General Singular Integral of Weierstrass 257	
6.4.2	Typical Means 261	
<b>6.5</b>	<b>Multipliers . . . . .</b>	<b>266</b>
6.5.1	Multipliers of Classes of Periodic Functions 266	
6.5.2	Multipliers on $L^p$ 268	
<b>6.6</b>	<b>Notes and Remarks . . . . .</b>	<b>273</b>
<b>7</b>	<b>Fourier Transform Methods and Second-Order Partial Differential Equations</b>	
<b>7.0</b>	<b>Introduction . . . . .</b>	<b>278</b>

7.1	Finite Fourier Transform Method . . . . .	281
7.1.1	Solution of Heat Conduction Problems	281
7.1.2	Dirichlet's and Neumann's Problem for the Unit Disc	284
7.1.3	Vibrating String Problems	287
7.2	Fourier Transform Method in $L^1$ . . . . .	294
7.2.1	Diffusion on an Infinite Rod	294
7.2.2	Dirichlet's Problem for the Half-Plane	297
7.2.3	Motion of an Infinite String	298
7.3	Notes and Remarks . . . . .	300
<b>Part III</b>		
<b>Hilbert Transforms . . . . .</b>		303
<b>8</b>	<b>Hilbert Transforms on the Real Line</b>	
8.0	Introduction . . . . .	305
8.1	Existence of the Transform . . . . .	307
8.1.1	Existence Almost Everywhere	307
8.1.2	Existence in $L^2$ -Norm	310
8.1.3	Existence in $L^p$ -Norm, $1 < p < \infty$	312
8.2	Hilbert Formulae, Conjugates of Singular Integrals, Iterated Hilbert Transforms . . . . .	316
8.2.1	Hilbert Formulae	316
8.2.2	Conjugates of Singular Integrals: $1 < p < \infty$	318
8.2.3	Conjugates of Singular Integrals: $p = 1$	320
8.2.4	Iterated Hilbert Transforms	323
8.3	Fourier Transforms of Hilbert Transforms . . . . .	324
8.3.1	Signum Rule	324
8.3.2	Summation of Allied Integrals	325
8.3.3	Fourier Transforms of Derivatives of Hilbert Transforms, the Classes $(W^\sim)_{[p}^r, (V^\sim)_{[p}^r$	327
8.3.4	Norm-Convergence of the Fourier Inversion Integral	329
8.4	Notes and Remarks . . . . .	331
<b>9</b>	<b>Hilbert Transforms of Periodic Functions</b>	
9.0	Introduction . . . . .	334
9.1	Existence and Basic Properties . . . . .	335
9.1.1	Existence	335
9.1.2	Hilbert Formulae	338
9.2	Conjugates of Singular Integrals . . . . .	341
9.2.1	The Case $1 < p < \infty$	341
9.2.2	Convergence in $C_{2\pi}$ and $L_{2\pi}^1$	341
9.3	Fourier Transforms of Hilbert Transforms . . . . .	347
9.3.1	Conjugate Fourier Series	347
9.3.2	Fourier Transforms of Derivatives of Conjugate Functions, the Classes $(W^\sim)_{x_{2\pi}}^r, (V^\sim)_{x_{2\pi}}^r$	349
9.3.3	Norm-Convergence of Fourier Series	350
9.4	Notes and Remarks . . . . .	353

<b>Part IV</b>	
<b>Characterization of Certain Function Classes . . . . .</b>	<b>355</b>
<b>10 Characterization in the Integral Case</b>	
10.0 Introduction . . . . .	357
10.1 Generalized Derivatives, Characterization of the Classes $W_{X_{2\pi}}^r$ . . . . .	358
10.1.1 Riemann Derivatives in $X_{2\pi}$ -Norm 358	
10.1.2 Strong Peano Derivatives 361	
10.1.3 Strong and Weak Derivatives, Weak Generalized Derivatives 363	
10.2 Characterization of the Classes $V_{X_{2\pi}}^r$ . . . . .	366
10.3 Characterization of the Classes $(V^\sim)_{X_{2\pi}}^r$ . . . . .	371
10.4 Relative Completion . . . . .	373
10.5 Generalized Derivatives in $L^p$ -Norm and Characterizations for $1 \leq p \leq 2$ . . . . .	376
10.6 Generalized Derivatives in $X(\mathbb{R})$ -Norm and Characterizations of the Classes $W_{X(\mathbb{R})}^r$ and $V_{X(\mathbb{R})}^r$ . . . . .	382
10.7 Notes and Remarks . . . . .	389
<b>11 Characterization in the Fractional Case</b>	
11.0 Introduction . . . . .	391
11.1 Integrals of Fractional Order . . . . .	393
11.1.1 Integral of Riemann–Liouville 393	
11.1.2 Integral of M. Riesz 396	
11.2 Characterizations of the Classes $W[L^p;  v ^\alpha]$ , $V[L^p;  v ^\alpha]$ , $1 \leq p \leq 2$ . . . . .	400
11.2.1 Derivatives of Fractional Order 400	
11.2.2 Strong Riesz Derivatives of Higher Order, the Classes $V[L^p;  v ^\alpha]$ 405	
11.3 The Operators $R_\delta^{(\alpha)}$ on $L^p$ , $1 \leq p \leq 2$ . . . . .	409
11.3.1 Characterizations 409	
11.3.2 Theorems of Bernstein–Titchmarsh and H. Weyl 414	
11.4 The Operators $R_\delta^{(\alpha)}$ on $X_{2\pi}$ . . . . .	416
11.5 Integral Representations, Fractional Derivatives of Periodic Functions . . . . .	419
11.6 Notes and Remarks . . . . .	428
<b>Part V</b>	
<b>Saturation Theory . . . . .</b>	<b>431</b>
<b>12 Saturation for Singular Integrals on <math>X_{2\pi}</math> and <math>L^p</math>, <math>1 \leq p \leq 2</math></b>	
12.0 Introduction . . . . .	433
12.1 Saturation for Periodic Singular Integrals, Inverse Theorems . . . . .	435
12.2 Favard Classes . . . . .	440
12.2.1 Positive Kernels 440	
12.2.2 Uniformly Bounded Multipliers 441	
12.2.3 Functional Equations 446	
12.3 Saturation in $L^p$ , $1 \leq p \leq 2$ . . . . .	452
12.3.1 Saturation Property 452	
12.3.2 Characterizations of Favard Classes: $p = 1$ 455	
12.3.3 Characterizations of Favard Classes: $1 < p \leq 2$ 459	

<b>12.4 Applications to Various Singular Integrals . . . . .</b>	<b>463</b>
12.4.1 Singular Integral of Fejér . . . . .	463
12.4.2 Generalized Singular Integral of Picard . . . . .	464
12.4.3 General Singular Integral of Weierstrass . . . . .	465
12.4.4 Singular Integral of Bochner–Riesz . . . . .	467
12.4.5 Riesz Means . . . . .	469
<b>12.5 Saturation of Higher Order . . . . .</b>	<b>471</b>
12.5.1 Singular Integrals on the Real Line . . . . .	471
12.5.2 Periodic Singular Integrals . . . . .	474
<b>12.6 Notes and Remarks . . . . .</b>	<b>478</b>
<b>13 Saturation on <math>X(\mathbb{R})</math></b>	
<b>13.0 Introduction . . . . .</b>	<b>483</b>
<b>13.1 Saturation of <math>D_\sigma(f; x; \tau)</math> in <math>X(\mathbb{R})</math>, Dual Methods . . . . .</b>	<b>485</b>
<b>13.2 Applications to Approximation in <math>L^p</math>, <math>2 &lt; p &lt; \infty</math> . . . . .</b>	<b>488</b>
13.2.1 Differences . . . . .	488
13.2.2 Singular Integrals Satisfying (12.3.5) . . . . .	489
13.2.3 Strong Riesz Derivatives . . . . .	490
13.2.4 The Operators $R_\varepsilon^{(\alpha)}$ . . . . .	491
13.2.5 Riesz and Fejér Means . . . . .	492
<b>13.3 Comparison Theorems . . . . .</b>	<b>493</b>
13.3.1 Global Divisibility . . . . .	493
13.3.2 Local Divisibility . . . . .	495
13.3.3 Special Comparison Theorems with no Divisibility Hypothesis . . . . .	498
13.3.4 Applications to Periodic Continuous Functions . . . . .	500
<b>13.4 Saturation on Banach Spaces . . . . .</b>	<b>502</b>
13.4.1 Strong Approximation Processes . . . . .	502
13.4.2 Semi-Groups of Operators . . . . .	504
<b>13.5 Notes and Remarks . . . . .</b>	<b>507</b>
<b>List of Symbols . . . . .</b>	<b>511</b>
<b>Tables of Fourier and Hilbert Transforms . . . . .</b>	<b>515</b>
<b>Bibliography . . . . .</b>	<b>521</b>
<b>Index</b>	<b>547</b>

