



## TABLE OF CONTENTS

### PART I. SIMPLER PROBLEMS OF THE CALCULUS OF VARIATIONS

CHAPTER	PAGE
I. THE CALCULUS OF VARIATIONS IN THREE-SPACE . . . . .	3
1. The nature of problems of the calculus of variations . . . . .	3
2. The origin of the name "calculus of variations" . . . . .	6
3. Analytic formulation of the problem . . . . .	7
4. The first and second variations . . . . .	9
5. The fundamental lemma . . . . .	10
6. Necessary conditions from the first variation . . . . .	11
7. Families of extremals . . . . .	15
8. Auxiliary theorems . . . . .	18
9. The necessary conditions of Weierstrass and Legendre . . . . .	20
10. Envelope theorems and Jacobi's necessary condition . . . . .	24
11. A second proof of Jacobi's condition . . . . .	27
12. The determination of conjugate points . . . . .	29
13. The geometric interpretation of conjugate points . . . . .	34
II. SUFFICIENT CONDITIONS FOR A MINIMUM . . . . .	37
14. Introduction . . . . .	37
15. Auxiliary theorems . . . . .	38
16. The sufficiency theorems of Weierstrass . . . . .	39
17. A comparison of necessary with sufficient conditions . . . . .	43
18. The definition and first properties of a field . . . . .	43
19. A fundamental sufficiency theorem . . . . .	45
20. Methods of constructing fields . . . . .	46
21. Sufficient conditions for an integral to be independent of the path . . . . .	49
22. Further properties of the slope functions and extremals of a field . . . . .	51
23. The theory of the second variation . . . . .	54
24. Sufficiency proofs without the use of fields . . . . .	59
III. FIELDS AND THE HAMILTON-JACOBI THEORY . . . . .	65
25. Introduction . . . . .	65
26. Canonical variables and canonical equations for extremals . . . . .	65
27. A second proof of the imbedding theorem . . . . .	68
28. Transversal surfaces of a field and the Hamilton-Jacobi equation . . . . .	70
29. Extremals as characteristics of a partial differential equation . . . . .	73
30. An application in dynamics . . . . .	76
31. Extremals as curves of quickest descent . . . . .	77
IV. PROBLEMS IN THE PLANE AND IN HIGHER SPACES . . . . .	81
32. Introduction . . . . .	81
33. The problem in the plane . . . . .	82

CHAPTER	PAGE
34. A comparison of the problems in the plane and in three-space . . . . .	83
35. The problem in a space of higher dimensions . . . . .	86
36. The determination of conjugate points . . . . .	88
37. The construction of fields . . . . .	89
38. The Hamilton-Jacobi theory . . . . .	92
39. The theory of the second variation . . . . .	97
<b>V. PROBLEMS IN PARAMETRIC FORM . . . . .</b>	<b>102</b>
40. Introduction . . . . .	102
41. Parametric representations of arcs . . . . .	103
42. Formulation of the parametric problem . . . . .	104
43. Consequences of the homogeneity relation . . . . .	105
44. First necessary conditions for a minimum . . . . .	108
45. The extremals . . . . .	111
46. The envelope theorem and Jacobi's condition . . . . .	116
47. Analytic proof of the condition of Jacobi . . . . .	118
48. The determination of conjugate points . . . . .	119
49. Fields and a fundamental sufficiency theorem . . . . .	124
50. Sufficient conditions for relative minima . . . . .	127
51. Further sufficient conditions for strong relative minima . . . . .	130
52. Canonical variables and equations . . . . .	132
53. The imbedding theorem and the Hamilton-Jacobi theory . . . . .	136
54. The construction of a complete integral of the Hamilton-Jacobi equation . . . . .	141
55. Other theories of parametric problems . . . . .	143
<b>VI. PROBLEMS WITH VARIABLE END-POINTS . . . . .</b>	<b>147</b>
56. Introduction . . . . .	147
57. Problems in three-space with one end-point variable on a surface . . . . .	148
58. Problems in three-space with one end-point variable on a curve . . . . .	152
59. A more general problem with variable end-points . . . . .	157
60. A sufficiency theorem for the more general problem with variable end- points . . . . .	158
61. The transversality condition . . . . .	162
62. The second variation and a fourth necessary condition . . . . .	163
63. Further sufficiency theorems . . . . .	166
64. Another form of the fourth condition . . . . .	167
65. Focal points for problems with one end-point variable . . . . .	170
66. Dependence of the focal point of a curve on curvature . . . . .	175
67. Problems with variable end-points in the plane . . . . .	180
<b>PART II. THE PROBLEM OF BOLZA</b>	
<b>VII. THE MULTIPLIER RULE . . . . .</b>	<b>187</b>
68. Introduction . . . . .	187
69. The equivalence of various problems . . . . .	189
70. Analytic formulation of the problem of Bolza . . . . .	193
71. Variations and the equations of variation . . . . .	194

# TABLE OF CONTENTS

ix

CHAPTER	PAGE
72. A fundamental imbedding lemma . . . . .	196
73. The first variation of $J$ . . . . .	199
74. The multiplier rule . . . . .	200
75. The extremals . . . . .	206
76. Abnormality for minima of functions of a finite number of variables	210
77. Normality for the problem of Bolza . . . . .	213
<b>VIII. FURTHER NECESSARY CONDITIONS FOR A MINIMUM . . . . .</b>	<b>220</b>
78. The necessary conditions of Weierstrass and Clebsch . . . . .	220
79. A lemma and a corollary . . . . .	224
80. The second variation and a fourth necessary condition for a minimum	226
81. The accessory minimum problem . . . . .	228
<b>IX. SUFFICIENT CONDITIONS FOR A MINIMUM . . . . .</b>	<b>235</b>
82. Statement of the sufficiency theorem . . . . .	235
83. An auxiliary theorem . . . . .	236
84. Fields and their construction . . . . .	237
85. A fundamental sufficiency theorem . . . . .	240
86. The second variation for problems with separated end-conditions satisfying also the non-tangency condition . . . . .	243
87. The sufficiency theorem for problems with separated end-conditions satisfying also the non-tangency condition . . . . .	247
88. Sufficiency theorems for problems with end-conditions unrestricted .	249
89. The second variation for problems with fixed end-points . . . . .	253
90. Conditions equivalent to the strengthened fourth condition . . . . .	257
91. Boundary-value problems associated with the second variation . . . . .	260
92. A sufficiency theorem applicable to some important abnormal cases .	264

## APPENDIX

<b>APPENDIX. EXISTENCE THEOREMS FOR IMPLICIT FUNCTIONS AND DIFFERENTIAL EQUATIONS . . . . .</b>	<b>269</b>
<b>I. EXISTENCE THEOREMS FOR IMPLICIT FUNCTIONS . . . . .</b>	<b>269</b>
1. The fundamental existence theorem for implicit functions . . . . .	269
2. An extension of the theorem of the preceding section . . . . .	272
<b>II. EXISTENCE THEOREMS FOR DIFFERENTIAL EQUATIONS . . . . .</b>	<b>274</b>
3. The existence of a solution through an initial point . . . . .	274
4. Existence theorem for linear equations . . . . .	276
5. The imbedding theorem . . . . .	276
6. Derivatives with respect to constants of integration . . . . .	278

## BIBLIOGRAPHY

<b>A BIBLIOGRAPHY FOR THE PROBLEM OF BOLZA . . . . .</b>	<b>285</b>
--	------------

## INDEX

<b>INDEX</b>	<b>291</b>
--------------	------------