Contents

PREFACE vii

CHAPTER 1. The Origin of Nonlinear Differential Equations

1.0 Introduction 1
1.1 What is Nonlinearity? 2
1.2 Equations from Diffusion Theory 4
1.3 Equations from Fluid Mechanics 8
1.4 Equations from Solid Mechanics 10
1.5 Miscellaneous Examples 13
1.6 Selected References 17
References 17

CHAPTER 2. Transformation and General Solutions

2.0 Introduction 20
2.1 Transformations on Dependent Variables 21
2.2 Transformations on Independent Variables 31
2.3 Mixed Transformations 35
2.4 The Unknown Function Approach 47
2.5 General Solutions 49
2.6 General Solutions of First-Order Equations 50
2.7 General Solutions of Second-Order Equations 58
2.8 Table of General Solutions 65
References 69

CHAPTER 3. Exact Methods of Solution

3.0 Introduction 71
3.1 The Quasi-Linear System 72
3.2 An Example of the Quasi-Linear Theory 78
3.3 The Poisson-Euler-Darboux Equation 84
3.4 Remarks on the PED Equation 88
3.5 One-Dimensional Anisentropic Flows 90
3.6 An Alternate Approach to Anisentropic Flow 94
3.7 General Solution for Anisentropic Flow 100
3.8 Vibration of a Nonlinear String 103
3.9 Other Examples of the Quasi-Linear Theory 109
3.10 Direct Separation of Variables 109
3.11 Other Solutions Obtained by *Ad Hoc* Assumptions 117
References 121

CHAPTER 4. Further Analytic Methods

4.0 Introduction 123
4.1 An *Ad Hoc* Solution from Magneto-Gas Dynamics 123
4.2 The Utility of Lagrangian Coordinates 126
4.3 Similarity Variables 133
4.4 Similarity via One-Parameter Groups 135
4.5 Extensions of the Similarity Procedure 141
4.6 Similarity via Separation of Variables 144
4.7 Similarity and Conservation Laws 150
4.8 General Comments on Transformation Groups 156
4.9 Similarity Applied to Moving Boundary Problems 158
4.10 Similarity Considerations in Three Dimensions 162
4.11 General Discussion of Similarity 166
4.12 Integral Equation Methods 167
4.13 The Hodograph 171
4.14 Simple Examples of Hodograph Application 173
4.15 The Hodograph in More Complicated Problems 177
4.16 Utilization of the General Solutions of Chapter 2 180
4.17 Similar Solutions in Heat and Mass Transfer 183
4.18 Similarity Integrals in Compressible Gases 186
4.19 Some Disjoint Remarks 190
References 192

CHAPTER 5. Approximate Methods

5.0 Introduction 195
5.1 Perturbation Concepts 196
5.2 Regular Perturbations in Vibration Theory 197
5.3 Perturbation and Plasma Oscillations 198
5.4 Perturbation in Elasticity 204
5.5 Other Applications 207
5.6 Perturbation about Exact Solutions 208
5.7 The Singular Perturbation Problem 211
5.8 Singular Perturbations in Viscous Flow 215
5.9 The "Inner-Outer" Expansion (a Motivation) 219
5.10 The Inner and Outer Expansions 222
5.11 Examples 226
5.12 Higher Approximations for Flow past a Sphere 231
5.13 Asymptotic Approximations 237
5.14 Asymptotic Solutions in Diffusion with Reaction 240
5.15 Weighted Residual Methods: General Discussion 243
5.16 Examples of the Use of Weighted Residual Methods 249
5.17 Comments on the Methods of Weighted Residuals 261
5.18 Mathematical Problems of Approximate Methods 262
References 267

CHAPTER 6. Further Approximate Methods

6.0 Introduction 271
6.1 Integral Methods in Fluid Mechanics 271
6.2 Nonlinear Boundary Conditions 278
6.3 Integral Equations and Boundary Layer Theory 280
6.4 Iterative Solutions for $\nabla^2 u = bu^2$ 284
6.5 The Maximum Operation 287
6.6 Equations of Elliptic Type and the Maximum Operation 289
6.7 Other Applications of the Maximum Operation 292
6.8 Series Expansions 295
6.9 Goertler's Series 299
6.10 Series Solutions in Elasticity 301
6.11 "Traveling Wave" Solutions by Series 305
References 312

CHAPTER 7. Numerical Methods

7.0 Introduction 315
7.1 Terminology and Computational Molecules 316

A. Parabolic Equations

7.2 Explicit Methods for Parabolic Systems 320
7.3 Some Nonlinear Examples 324
7.4 Alternate Explicit Methods 326
7.5 The Quasi-Linear Parabolic Equation 330
7.6 Singularities 330
7.7 A Treatment of Singularities (Example) 334
7.8 Implicit Procedures 338
7.9 A Second-Order Method for $Lu = f(x, t, u)$ 343
7.10 Predictor Corrector Methods 345
7.11 Traveling Wave Solutions 348
7.12 Finite Differences Applied to the Boundary Layer Equations 349
7.13 Other Nonlinear Parabolic Examples 355

B. Elliptic Equations

7.14 Finite Difference Formula for Elliptic Equations in Two Dimensions 365
7.15 Linear Elliptic Equations 370
7.16 Methods of Solution of $Au = v$ 373
7.17 Point Iterative Methods 375
7.18 Block Iterative Methods 384
7.19 Examples of Nonlinear Elliptic Equations 389
7.20 Singularities 411

C. Hyperbolic Equations

7.21 Method of Characteristics 416
7.22 The Supersonic Nozzle 423
7.23 Properties of Hyperbolic Systems 426

7.24 One-Dimensional Isentropic Flow 432
7.25 Method of Characteristics: Numerical Computation 435
7.26 Finite Difference Methods: General Discussion 437
7.27 Explicit Methods 438
7.28 Explicit Methods in Nonlinear Second-Order Systems 440
7.29 Implicit Methods for Second-Order Equations 443
7.30 "Hybrid" Methods for a Nonlinear First-Order System 445
7.31 Finite Difference Schemes in One-Dimensional Flow 448
7.32 Conservation Equations 453
7.33 Interfaces 454
7.34 Shocks 456
7.35 Additional Methods 461

D. Mixed Systems

7.36 The Role of Mixed Systems 462
7.37 Hydrodynamic Flow and Radiation Diffusion 462
7.38 Nonlinear Vibrations of a Moving Threadline 464
References 467

CHAPTER 8. Some Theoretical Considerations

8.0 Introduction 474
8.1 Well-Posed Problems 475
8.2 Existence and Uniqueness in Viscous Incompressible Flow 478
8.3 Existence and Uniqueness in Boundary Layer Theory 482
8.4 Existence and Uniqueness in Quasi-Linear Parabolic Equations 486
8.5 Uniqueness Questions for Quasi-Linear Elliptic Equations 487
References 489

APPENDIX. Elements of Group Theory

A.1 Basic Definitions 491
A.2 Groups of Transformations 492

AUTHOR INDEX 495
SUBJECT INDEX 501