Contents

Preface xi

List of symbols xiii

1 Introduction 1

2 General results on the forward stability of recursion relations 4
2.1 Background 4
2.2 Homogeneous systems 6
2.3 Nonhomogeneous systems 11
2.4 The first-order scalar case; forward vs. backward recursion 12
2.5 The computation of successive derivatives 14
2.6 Scalar equations of higher order: minimal and dominant solutions 17

3 First-order equations used in the backward direction: the Miller algorithm 23
3.1 Introduction: the algorithm 23
3.2 Convergence and error analysis 25

4 Second-order homogeneous equations: the Miller algorithm 29
4.1 The algorithm 29
4.2 Reduction of the error by the use of asymptotic information 37
4.3 Error analysis, the simplified algorithm: case of negative coefficients 38
4.4 Error analysis, the general algorithm 45
4.5 Algorithms based on continued fractions 46
4.6 Minimal solutions and orthogonal polynomials 53
4.7 The Clenshaw averaging process 57

5 Applications of the Miller algorithm to the computation of special functions 61
5.1 The confluent hypergeometric function $\Phi(a, c; x)$ 61
5.2 The confluent hypergeometric function $\Psi(a, c; x)$ 63

5.3 The Gaussian hypergeometric function 70
5.4 Associated Legendre functions 73
5.5 The Legendre function $Q_\nu^\mu(z)$ 75
5.6 The Jacobi polynomials $P_n^{(i\eta,-i\eta)}(-i\omega)$ 75
5.7 Bessel functions 78
5.8 Zeros of Bessel functions 82
5.9 Eigenvalues of Mathieu's equation 82

6 Second-order nonhomogeneous equations: the Olver algorithm 86
6.1 Introduction: the algorithm 86
6.2 Solution by forward elimination (Method A) 90
6.3 The method of averaging (Method B) 94
6.4 The LU-decomposition (Method C) 96
6.5 Adaptation to general normalizing conditions 98
6.6 Conclusions 103

7 Higher-order systems: homogeneous equations 104
7.1 Observations on higher-order equations 104
7.2 The Miller algorithm 105
7.3 The matrix formulation: stability and weak stability 114
7.4 The Clenshaw averaging process 122
7.5 Topics for future research: infinite systems 128
7.5.1 Basic series for functions satisfying functional equations 128
7.5.2 Stieltjes moment integrals 131

8 Higher-order systems (continued): the nonhomogeneous case 135
8.1 Introduction 135
8.2 The Wimp–Luke method 137
8.3 The Lozier algorithm 139

9 The computation of ${}_3F_2(1)$ 152
9.1 The recursion 152
9.2 The algorithm; truncation error 154
9.3 Computing the Beta function 157
9.4 Another ${}_3F_2(1)$ 158

10 Computations based on orthogonal polynomials 161
10.1 Preliminaries: properties of some orthogonal polynomials 161
10.1.1 Chebyshev polynomials 161
10.1.2 Jacobi polynomials 163
10.2 Evaluation of finite sums of functions which satisfy a linear homogeneous recurrence 165

10.2.1 The algorithm 165
10.2.2 Error analysis; three-term recurrence 167
10.2.3 Converting one expansion into another 172

11 Series solutions to ordinary differential equations 177
11.1 Taylor series solutions 178
11.2 The construction of general recurrence relations for the coefficients of Gegenbauer series 186
11.2.1 Introduction and basic formulas 186
11.2.2 The algorithms of Clenshaw and Elliott 188
11.2.3 The Lewanowicz construction 196
11.3 Chebyshev series solutions for nonlinear differential equations 202

12 Multidimensional recursion algorithms; general theory 209
12.1 Background: convergence properties of complex sequences 209
12.2 Introduction; invariants 211
12.3 Divergence; strange attractors 219
12.4 Mean values 223

13 Two-dimensional algorithms 229
13.1 General remarks: invariant curves 229
13.2 Evaluation of certain infinite products 231
13.3 Carlson's results 237
13.4 An algorithm of Gatteschi 245
13.5 Tricomi's algorithm 248
13.6 Solutions of linear functional equations 249

14 Higher-dimensional algorithms 254
14.1 The computation of a class of trigonometric integrals 254
14.2 Incomplete elliptic integrals 260

Appendix A The general theory of linear difference equations 265

Appendix B The asymptotic theory of linear difference equations 270
B.1 General theory 270
B.2 The construction of formal series solutions 272
B.3 The Olver growth theorems 282

Appendix C Recursion formulas for hypergeometric functions 289

Index of higher mathematical functions discussed 293

Bibliography 295

Index 307