



# CONTENTS

Introduction . . . . .	1
PART I. THE MODULUS OF SMOOTHNESS	
Chapter 1. Preliminaries . . . . .	7
1.1. Notations . . . . .	7
1.2. Discussion of Some Conditions on $\varphi(x)$ . . . . .	8
1.3. Examples of Various Step-Weight Functions $\varphi(x)$ . . . . .	9
Chapter 2. The $K$ -Functional and the Modulus of Continuity . . . . .	10
2.1. The Equivalence Theorem . . . . .	10
2.2. The Upper Estimate, $K_{r,\varphi}(f,t')_p \leq M\omega_\varphi^r(f,t)_p$ , Case I . . . . .	12
2.3. The Upper Estimate of the $K$ -Functional, The Other Cases . . . . .	16
2.4. The Lower Estimate for the $K$ -Functional . . . . .	20
Chapter 3. $K$ -Functionals and Moduli of Smoothness, Other Forms .	24
3.1. A Modified $K$ -Functional . . . . .	24
3.2. Forward and Backward Differences . . . . .	26
3.3. Main-Part Modulus of Smoothness . . . . .	28
3.4. Computation of Our Modulus for Some Functions . . . . .	34
Chapter 4. Properties of $\omega_\varphi^r(f,t)_p$ . . . . .	36
4.1. Extending the Basic Properties of the Classical Moduli . . . . .	36
4.2. Optimal Rate of $\omega_\varphi^r(f,t)$ . . . . .	40
4.3. Marchaud Inequality . . . . .	43

<b>Chapter 5. More General Step-Weight Functions <math>\varphi</math></b> . . . . .	<b>46</b>
5.1. Logarithmic-Type Weights and Internal Zeros . . . . .	46
5.2. The Necessity of the Finite Overlapping Condition . . . . .	47
5.3. Growth Order of Type $x^\beta$ with Arbitrary $\beta$ . . . . .	49
<b>Chapter 6. Weighted Moduli of Smoothness. . . . .</b>	<b>55</b>
6.1. Weighted Moduli of Smoothness and Weighted $K$ -Functionals . . . . .	55
6.2. The Weighted Main-Part Modulus . . . . .	59
6.3. Smoothness Properties of Derivatives . . . . .	62
6.4. Marchaud Inequality for Weighted Main-Part Moduli . . . . .	67
6.5. Connection with Ordinary Weighted Moduli. . . . .	71
 <b>PART II. APPLICATIONS</b>	
<b>Chapter 7. Algebraic Polynomial Approximation . . . . .</b>	<b>77</b>
7.1. Background. . . . .	77
7.2. Best Polynomial Approximation . . . . .	79
7.3. Asymptotic Behavior of Derivatives of Best Approximating Polynomials. . . . .	84
7.4. Error Bounds for Gaussian Quadrature. . . . .	87
<b>Chapter 8. Weighted Best Polynomial Approximation . . . . .</b>	<b>90</b>
8.1. Some Concepts and Description of the Weight . . . . .	90
8.2. Best Weighted Algebraic Polynomial Approximation . . . . .	94
8.3. Derivatives of the Optimal Polynomials. . . . .	98
8.4. Proof of Some Crucial Inequalities for $w \in J_p^*$ . . . . .	100
8.5. Applications, Calculations, and Specific Examples . . . . .	109
<b>Chapter 9. Exponential-Type or Bernstein-Type Operators . . . . .</b>	<b>112</b>
9.1. Background and Notations, Positive Operators on $C(D)$ . . . . .	112
9.2. Operators on $L_p(D)$ , Higher Degree of Smoothness . . . . .	115
9.3. Direct and Converse Results . . . . .	117
9.4. The Bernstein-Type Inequality $\ \varphi^{2r} L_n^{(2r)} f\ _p \leq M n^r \ f\ _p$ . . . . .	124
9.5. Rate of Convergence for Smooth Functions . . . . .	134
9.6. Estimate of $\ L_n(R_{2r}(f, \cdot, x), x)\ _{L_p(E_n)}$ . . . . .	140
9.7. The Estimate $\ \varphi(x)^{2r} L_n^{(2r)}(f)\ _{L_p} \leq M \ \varphi^{2r} f^{(2r)}\ _p$ . . . . .	152
<b>Chapter 10. Weighted Approximations by Exponential-Type Operators</b> . . . . .	<b>158</b>
10.1. The Direct and Inverse Result . . . . .	158
10.2. The Boundedness of the Operators in Weighted Norm. . . . .	161
10.3. Bernstein-Type Inequality . . . . .	165
10.4. The Estimate $\ w \varphi^2 L_n^{(2)}(f)\  \leq C(\ w \varphi^2 f^{(2)}\  + \ f\ )$ . . . . .	168
10.5. The Estimate of $L_n f - f$ for Smooth Functions . . . . .	168
10.6. The Saturation Result . . . . .	175

<b>Chapter 11. Weighted Polynomial Approximation in <math>L_p(R)</math></b> . . . . .	180
11.1. Introduction . . . . .	180
11.2. The Equivalence Result . . . . .	181
11.3. The Direct and Converse Results . . . . .	184
11.4. Proof of the Equivalence Result . . . . .	186
11.5. Comparisons and Generalizations . . . . .	195
<b>Chapter 12. Polynomial Approximation in Several Variables</b> . . . . .	197
12.1. Approximation on Cubes . . . . .	197
12.2. Approximation on Polytopes . . . . .	201
<b>Chapter 13. Comparisons and Conclusions</b> . . . . .	211
13.1. Comparison with Similar Expressions . . . . .	211
13.2. The Integral Modulus of Smoothness of Ivanov and Sendov . . . . .	212
13.3. Moduli Generated by Multipliers and Integral Transforms . . . . .	213
13.4. A Modulus Introduced by Potapov . . . . .	214
13.5. Hoeffding's Result . . . . .	215
13.6. Conclusion . . . . .	216
<b>Appendix</b> . . . . .	217
A. The Analogue of Definition 5.3.1 . . . . .	217
B. The Definition of the Weighted Modulus of Smoothness on $(0, 1)$ . . . . .	218
<b>References</b> . . . . .	219
<b>List of Symbols</b> . . . . .	226
<b>Index</b> . . . . .	227