

Contents

Preface	v
Hints for Reading the Book	vi
Notations	xix
1. Introduction	1
1.1 Historical Remarks Concerning Iterative Methods	1
1.2 Model Problem (Poisson Equation)	2
1.3 Amount of Work for the Direct Solution of the System of Equations	5
1.4 Examples of Iterative Methods	7
2. Recapitulation of Linear Algebra	12
2.1 Notations for Vectors and Matrices	12
2.1.1 Nonordered Index Sets	12
2.1.2 Notations	13
2.1.3 Star Notation	14
2.2 Systems of Linear Equations	16
2.3 Permutation Matrices	16
2.4 Eigenvalues and Eigenvectors	17
2.5 Block-Vectors and Block-Matrices	21
2.6 Norms	23
2.6.1 Vector Norms	23
2.6.2 Equivalence of All Norms	24
2.6.3 Corresponding Matrix Norms	25
2.7 Scalar Product	26
2.8 Normal Forms	28
2.8.1 Schur Normal Form	28
2.8.2 Jordan Normal Form	29
2.8.3 Diagonalisability	31

2.9	Correlation Between Norms and the Spectral Radius	33
2.9.1	Corresponding Matrix Norms as Upper Bound for the Eigenvalues	33
2.9.2	Spectral Norm	33
2.9.3	Matrix Norm Approximating the Spectral Radius	34
2.9.4	Geometrical Sum (Neumann's Series) for Matrices	36
2.9.5	Numerical Radius of a Matrix	36
2.10	Positive Definite Matrices	38
2.10.1	Definition and Notations	38
2.10.2	Rules and Criteria for Positive Definite Matrices	38
2.10.3	Remarks Concerning Positive Definite Matrices	39
3.	Iterative Methods	43
3.1	General Statements Concerning Convergence	43
3.1.1	Notations	43
3.1.2	Fixed Points	44
3.1.3	Consistency	44
3.1.4	Convergence	44
3.1.5	Convergence and Consistency	45
3.2	Linear Iterative Methods	46
3.2.1	Notations, First Normal Form	46
3.2.2	Consistency, Second and Third Normal Form	46
3.2.3	Representation of the Iterates x^m	47
3.2.4	Convergence	48
3.2.5	Convergence Speed	50
3.2.6	Remarks Concerning the Matrices M , N , and W	52
3.2.7	Product Iterations	53
3.2.8	Three-Term Recursions (Two-Step Iterations)	54
3.3	Effectiveness of Iterative Methods	54
3.3.1	Amount of Computational Work	55
3.3.2	Effectiveness	55
3.3.3	Order of the Linear Convergence	56
3.4	Test of Iterative Methods	57
3.5	Comments Concerning the PASCAL Procedures	58
3.5.1	PASCAL	58
3.5.2	Concerning the Test Examples	59
3.5.3	Constants and Types	60
3.5.4	Format of the Iteration Procedures	62
3.5.5	Test Environment	62
4.	Methods of Jacobi and Gauß-Seidel and SOR Iteration in the Positive Definite Case	65
4.1	Eigenvalue Analysis of the Model Problem	65
4.2	Construction of Iterative Methods	67
4.2.1	Jacobi Iteration	67

4.2.1.1 Additive Splitting of the Matrix A	67
4.2.1.2 Definition of the Jacobi Method	68
4.2.1.3 PASCAL Procedure	69
4.2.2 Gauß-Seidel Method	70
4.2.2.1 Definition	70
4.2.2.2 PASCAL Procedure	71
4.3 Damped Iterative Methods	73
4.3.1 Damped Jacobi Method	73
4.3.1.1 Damping of a General Iterative Method	73
4.3.1.2 PASCAL Procedures	74
4.3.2 Richardson Iteration	75
4.3.2.1 Definition	75
4.3.2.2 PASCAL Procedures	76
4.3.3 SOR Method	77
4.3.3.1 Definition	77
4.3.3.2 PASCAL Procedures	78
4.4 Convergence Analysis	82
4.4.1 Richardson Iteration	82
4.4.2 Jacobi Iteration	87
4.4.3 Gauß-Seidel and SOR Methods	90
4.5 Block Versions	96
4.5.1 Block-Jacobi Method	96
4.5.1.1 Definition	96
4.5.1.2 PASCAL Procedures	97
4.5.2 Block-Gauß-Seidel and Block-SOR Method	100
4.5.2.1 Definition	100
4.5.2.2 PASCAL Procedures	101
4.5.3 Convergence of the Block Variants	104
4.6 Computational Work of the Methods	105
4.6.1 Case of General Sparse Matrices	105
4.6.2 Amount of Work in the Model Case	106
4.7 Convergence Rates in the Case of the Model Problem	107
4.7.1 Richardson and Jacobi Iteration	107
4.7.2 Block-Jacobi Iteration	108
4.7.3 Numerical Examples for the Jacobi Variants	109
4.7.4 SOR and Block-SOR Iteration with Numerical Examples	110
4.8 Symmetric Iterations	112
4.8.1 General Form of the Symmetric Iteration	112
4.8.2 Convergence	113
4.8.3 Symmetric Gauß-Seidel Method	114
4.8.4 Adjoint and Corresponding Symmetric Iterations	115
4.8.5 SSOR: Symmetric SOR	117
4.8.6 PASCAL Procedures and Numerical Results for the SSOR Method	119

5. Analysis in the 2-Cyclic Case	122
5.1 2-Cyclic Matrices	122
5.2 Preparatory Lemmata	125
5.3 Analysis of the Richardson Iteration	127
5.4 Analysis of the Jacobi Method	129
5.5 Analysis of the Gauß-Seidel Iteration	130
5.6 Analysis of the SOR Method	131
5.6.1 Consistently Ordered Matrices	131
5.6.2 Theorem of Young	134
5.6.3 Order Improvement by SOR	137
5.6.4 Practical Handling of the SOR Method	138
5.7 Application to the Model Problem	138
5.7.1 Analysis in the Model Case	138
5.7.2 Gauß-Seidel Iteration: Numerical Examples	140
5.7.3 SOR Iteration: Numerical Examples	140
5.8 Supplementary Remarks	141
5.8.1 p -Cyclic Matrices	141
5.8.2 Modified SOR	142
5.8.3 SSOR in the 2-Cyclic Case	142
5.8.4 Unsymmetric SOR Method	143
6. Analysis for M-Matrices	144
6.1 Positive Matrices	144
6.2 Graph of a Matrix and Irreducible Matrices	145
6.3 Perron-Frobenius Theory of Positive Matrices	148
6.4 M-Matrices	152
6.4.1 Definition	152
6.4.2 Connection Between M-Matrices and Jacobi Iteration	152
6.4.3 Diagonal Dominance	154
6.4.4 Further Criteria	156
6.5 Regular Splittings	159
6.6 Applications	162
7. Semi-Iterative Methods	165
7.1 First Formulation	165
7.1.1 The Semi-Iterative Sequence	165
7.1.2 Consistency and Asymptotical Convergence Rate	166
7.1.3 Error Representation	166
7.2 Second Formulation of a Semi-Iterative Method	167
7.2.1 General Representation	167
7.2.2 PASCAL Implementation of the Second Formulation	169
7.2.3 Three-Term Recursion	169
7.3 Optimal Polynomials	170
7.3.1 Minimisation Problem	170
7.3.2 Discussion of the Second Minimisation Problem	171

7.3.3	Chebyshev Polynomials	173
7.3.4	Chebyshev Method	173
7.3.5	Order Improvement by the Chebyshev Method	177
7.3.6	Optimisation over Other Sets	178
7.3.7	Cyclic Iteration	179
7.3.8	Reformulation	180
7.3.9	Multi-Step Iterations	181
7.3.10	PASCAL Procedures	182
7.3.11	Amount of Work of the Semi-Iterative Method	185
7.4	Application to Iterations Discussed Above	186
7.4.1	Preliminaries	186
7.4.2	Semi-Iterative Richardson Method	187
7.4.3	Semi-Iterative Jacobi and Block-Jacobi Method	188
7.4.4	Semi-Iterative SSOR and Block-SSOR Method	190
7.5	Method of Alternating Directions (ADI)	194
7.5.1	Application to the Model Problem	194
7.5.2	General Representation	195
7.5.3	ADI Method in the Commutative Case	197
7.5.4	ADI Method and Semi-Iterative Methods	201
7.5.5	PASCAL Procedures	202
7.5.6	Amount of Work and Numerical Examples	204

8. Transformations, Secondary Iterations, Incomplete Triangular Decompositions

		205
8.1	Generation of Iterations by Transformations	205
8.1.1	Already Discussed Techniques for Generating, Iterations	205
8.1.2	Left Transformation	206
8.1.3	Right Transformation	208
8.1.4	Two-Sided Transformation	209
8.2	Kaczmarz Iteration	210
8.2.1	Original Formulation	210
8.2.2	Interpretation as Gauß-Seidel Method	210
8.2.3	PASCAL Procedures and Numerical Examples	211
8.3	Preconditioning	212
8.3.1	Meaning of «Preconditioning»	212
8.3.2	Examples	214
8.3.3	Rules of Calculation for Condition Numbers	216
8.4	Secondary Iterations	217
8.4.1	Examples of Secondary Iterations	217
8.4.2	Convergence Analysis in the General Case	220
8.4.3	Analysis in the Symmetric Case	222
8.4.4	Estimate of the Amount of Work	224
8.4.5	PASCAL Procedures	226
8.4.6	Numerical Examples	226

8.5	Incomplete Triangular Decompositions	228
8.5.1	Introduction and ILU Iteration	228
8.5.2	Incomplete Decomposition with Respect to a Star Pattern	231
8.5.3	Application to General Five-Point Formulae	231
8.5.4	Modified ILU Decompositions	233
8.5.5	On the Existence and Stability of the ILU Decomposition	234
8.5.6	Properties of the ILU Decomposition	237
8.5.7	ILU Decompositions Corresponding to Other Patterns	240
8.5.8	Approximative ILU Decompositions	241
8.5.9	Blockwise ILU Decompositions	242
8.5.10	PASCAL Procedures	242
8.5.11	Numerical Examples	245
8.5.12	Comments	245
8.6	A Superfluous Term: Time-Stepping Methods	246
9.	Conjugate Gradient Methods	248
9.1	Linear Systems of Equations as Minimisation Problem	248
9.1.1	Minimisation Problem	248
9.1.2	Search Directions	249
9.1.3	Other Quadratic Functionals	250
9.1.4	Complex Case	251
9.2	Gradient Method	251
9.2.1	Construction	251
9.2.2	Properties of the Gradient Method	252
9.2.3	Numerical Examples	253
9.2.4	Gradient Method Based on Other Iterations	255
9.2.5	PASCAL Procedures and Numerical Examples	258
9.3	The Method of the Conjugate Directions	262
9.3.1	Optimality with Respect to a Direction	262
9.3.2	Conjugate Directions	264
9.4	Conjugate Gradient Method (cg Method)	266
9.4.1	First Formulation	266
9.4.2	cg Method (Applied to the Richardson Iteration)	269
9.4.3	Convergence Analysis	270
9.4.4	cg Method Applied to Symmetric Iterations	273
9.4.5	PASCAL Procedures	275
9.4.6	Numerical Examples in the Model Case	276
9.4.7	Amount of Work of the cg Method	278
9.4.8	Suitability for Secondary Iterations	279
9.5	Generalisations	280
9.5.1	Formulation of the cg Method with a More General Bilinear Form	280

9.5.2	Method of Conjugate Residuals	282
9.5.3	Three-Term Recursion for p^m	284
9.5.4	Stabilised Method of the Conjugate Residuals	286
9.5.5	Convergence Results for Indefinite Matrices A	286
9.5.6	PASCAL Procedures	289
9.5.7	Numerical Examples	290
9.5.8	Method of Orthogonal Directions	291
9.5.9	Solution of Unsymmetric Systems	294
9.5.10	Further Comments	295
10.	Multi-Grid Methods	296
10.1	Introduction	296
10.1.1	Smoothing	296
10.1.2	Hierarchy of Systems of Equations	299
10.1.3	Prolongation	299
10.1.4	Restriction	301
10.1.5	Coarse-Grid Correction	302
10.2	Two-Grid Method	304
10.2.1	Algorithm	304
10.2.2	Modifications	304
10.2.3	Iteration Matrix	304
10.2.4	PASCAL Procedures	305
10.2.5	Numerical Examples	311
10.3	Analysis for a One-Dimensional Example	312
10.3.1	Fourier Analysis	313
10.3.2	Transformed Quantities	314
10.3.3	Convergence Results	315
10.4	Multi-Grid Iteration	317
10.4.1	Algorithm	317
10.4.2	PASCAL Procedures	318
10.4.3	Numerical Examples	323
10.4.4	Computational Work	325
10.4.5	Iteration Matrix	328
10.5	Nested Iteration	328
10.5.1	Algorithm	328
10.5.2	Error Analysis	329
10.5.3	Amount of Computational Work	330
10.5.4	PASCAL Procedures	331
10.5.5	Numerical Examples	335
10.5.6	Comments	336
10.6	Convergence Analysis	336
10.6.1	Summary	336
10.6.2	Smoothing Property	337
10.6.3	Approximation Property	342
10.6.3.1	Formulation	342

10.6.3.2	Galerkin Discretisation	343
10.6.3.3	Hierarchy of the Systems of Equations	343
10.6.3.4	Canonical Prolongation and Restriction	345
10.6.3.5	Error Estimate of the Galerkin Solution	345
10.6.3.6	Proof of the Approximation Property	346
10.6.4	Convergence of the Two-Grid Iteration	347
10.6.5	Convergence of the Multi-Grid Iteration	348
10.6.6	Case of Weaker Regularity	349
10.7	Symmetric Multi-Grid Methods	350
10.7.1	Symmetric Multi-Grid Algorithm	350
10.7.2	Two-Grid Convergence for $\nu_1 > 0, \nu_2 > 0$	351
10.7.3	Smoothing Property in the Symmetric Case	352
10.7.4	Strengthened Two-Grid Convergence Estimates	353
10.7.5	V-Cycle Convergence	356
10.7.6	Multi-Grid Convergence for All $\nu > 0$	357
10.8	Combination of Multi-Grid Methods with Semi-Iterations	359
10.8.1	Semi-Iterative Smoothers	359
10.8.2	Damped Coarse-Grid Corrections	361
10.8.3	Multi-Grid Iteration as Basis of the Conjugate Gradient Method	361
10.9	Further Comments	362
10.9.1	Multi-Grid Method of the Second Kind	362
10.9.2	History of the Multi-Grid Method	362
10.9.3	Robust Methods	363
10.9.4	Frequency Filtering Decompositions	364
11.	Domain Decomposition Methods	367
11.1	Introduction	367
11.2	Formulation of the Domain Decomposition Method	368
11.2.1	General Construction	368
11.2.2	The Prolongations	370
11.2.3	Multiplicative and Additive Schwarz Iteration	371
11.2.4	Interpretation as Gauß-Seidel and Jacobi Iteration	372
11.2.5	Classical Schwarz Iteration	372
11.2.6	Approximate Solution of the Subproblems	373
11.2.7	Strengthened Estimate $A \leq \Gamma W$	374
11.3	Properties of the Additive Schwarz Iteration	376
11.3.1	Parallelism	376
11.3.2	Condition Estimates	377
11.3.3	Convergence Statements	379
11.4	Analysis of the Multiplicative Schwarz Iteration	381
11.4.1	Convergence Statements	381
11.4.2	Proofs of the Convergence Theorems	384
11.5	Examples	387
11.5.1	Schwarz Methods with Proper Domain Decomposition	387

11.5.2 Additive Schwarz Iteration with Coarse-Grid Correction	389
11.5.3 Formulation in the Case of a Galerkin Discretisation	389
11.6 Multi-Grid Methods as Subspace Decomposition Method ..	391
11.6.1 The Analysis of Braess	391
11.6.2 V-Cycle Interpreted as Multiplicative Schwarz Iteration	393
11.6.3 Proof of the V-Cycle Convergence	395
11.6.4 Method of the Hierarchical Basis	397
11.6.5 Multi-Level Schwarz Iteration	399
11.6.6 Further Approaches for Decompositions into Subspaces	400
11.6.7 Indefinite and Unsymmetric Systems	400
11.7 Schur Complement Methods	400
11.7.1 Nonoverlapping Domain Decomposition with Interior Boundary	400
11.7.2 Direct Solution	401
11.7.3 Capacitance Matrix Method	402
11.7.4 Domain Decomposition Method with Nonoverlapping Domains	402
11.7.5 Multi-Gridlike Domain Decomposition Methods ...	403
11.7.6 Further Remarks	404
Bibliography	405
Subject Index	418
Index of PASCAL Terms	426