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Those sections marked with “*” contain material of a rather specialized nature that may be omitted on first reading.

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