Contents

1. HISTORICAL INTRODUCTION 1

2. STRUCTURAL BACKGROUND 6
   - 2.1. Stiffness analysis for simple spring systems 6
   - 2.2. The principle of virtual displacements and element stiffness matrices 12
   - 2.3. Finite element idealization of simple structures 20
   - 2.4. Derivation of field variables (stresses) from the calculated displacements 27
   - 2.5. The relationship between the principle of virtual displacements and the principle of minimum potential energy 31
   - 2.6. Exercises and solutions 34

3. VARIATIONAL METHODS 43
   - 3.1. Classification of differential operators 43
   - 3.2. Self-adjoint positive definite operators 45
   - 3.3. The extremum formulation with homogeneous boundary conditions 48
   - 3.4. Non-homogeneous boundary conditions: Dirichlet, Neumann, and mixed 54
   - 3.5. The general second-order linear partial differential equation; natural boundary conditions 58
   - 3.6. The Rayleigh–Ritz method 61
   - 3.7. Functional for elasticity problems and the 'elastic analogy' for Poisson's equation 71
   - 3.8. Variational methods for time-dependent problems 75
   - 3.9. Weighted residual methods: collocation, least squares, and Galerkin 78
   - 3.10. Exercises and solutions 86

4. FINITE ELEMENT IDEALIZATION FOR FIELD PROBLEMS 105
   - 4.1. Difficulties associated with the application of the variational method 105
   - 4.2. Piecewise application of the Rayleigh–Ritz method 106
   - 4.3. Terminology 108
   - 4.4. Finite element idealization 109
   - 4.5. Illustrative problem involving one independent variable 115
   - 4.6. Finite element equations for Poisson's equation 126

4.7. A rectangular element for Poisson's equation 136
4.8. A triangular element for Poisson's equation 142
4.9. Exercises and solutions 151

5. HIGHER-ORDER ELEMENTS AND THE ISOPARAMETRIC CONCEPT 176
5.1. A two-point boundary-value problem 176
5.2. Higher-order rectangular elements 179
5.3. Higher-order triangular elements 180
5.4. Elements with more than one degree of freedom at each node 182
5.5. Condensation of internal nodal freedoms 186
5.6. Curved boundaries and higher-order elements: isoparametric elements 188
5.7. Exercises and solutions 195

6. FURTHER TOPICS IN THE FINITE ELEMENT METHOD 206
6.1. Collocation and least-squares methods 206
6.2. Galerkin's method: equivalence with the variational method 209
6.3. Use of Galerkin's method for time-dependent and non-linear problems 214
6.4. Time-dependent problems using variational principles which are not extremal: Laplace transform 225
6.5. Exercises and solutions 232

7. CONVERGENCE OF THE FINITE ELEMENT METHOD 248
7.1. A one-dimensional example 248
7.2. Two-dimensional problems involving Poisson's equation 254
7.3. Isoparametric elements: numerical integration 256
7.4. Non-conforming elements: the patch test 259
7.5. Comparison with the finite difference method: stability 260
7.6. Exercises and solutions 265

APPENDIX 1: Some integral theorems of the vector calculus 275
APPENDIX 2: A formula for integrating products of area coordinates over a triangle 277
APPENDIX 3: Numerical integration formulae 279
REFERENCES 281
INDEX 285