CONTENTS

Both section and page numbers are given for the topics below. The reader is recommended to use section numbers (at the top of the pages of the text proper) when looking up a given topic, since a better idea of the structure of the book will be obtained.

In general, review questions and/or problems follow after each one-decimal sub-section.

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