



# Contents

Preface 5

Introduction 11

## *Part I*

NUMERICAL METHODS OF MATHEMATICAL ANALYSIS 17

1 *The error of a numerical solution* 19

- 1.1 Sources and classification of errors 19
- 1.2 Number representation in computers 22
- 1.3 Absolute and relative errors. Notation 23
- 1.4 Computational error 25
- 1.5 The error of a function 26

2 *Interpolation and allied problems* 33

- 2.1 Approximation of functions. Statement of the problem 34
- 2.2 Lagrange's interpolation polynomial 38
- 2.3 Estimate of the remainder of Lagrange's interpolation polynomial 40
- 2.4 Divided differences and their properties 41
- 2.5 Newton's interpolation formula with divided differences 42
- 2.6 Divided differences and interpolation with multiple points 46
- 2.7 Difference equations 50
- 2.8 Chebyshev polynomials 60
- 2.9 Minimizing the remainder estimate of an interpolation formula 63
- 2.10 Finite differences 66
- 2.11 Newton's interpolation formulas for equal intervals 69
- 2.12 Interpolation formulas of Bessel and Everett. Compiling tables 71
- 2.13 Rounding errors in interpolation 80
- 2.14 Applying the apparatus of interpolation. Inverse interpolation 82
- 2.15 Orthogonal systems and their properties 83
- 2.16 Orthogonal polynomials 89
- 2.17 Numerical differentiation 93
- 2.18 On the computational error of formulas of numerical differentiation 97

3 *Numerical integration* 100

- 3.1 Newton-Cotes quadrature formulas 100
- 3.2 Error estimate of a quadrature formula on a class of functions 109
- 3.3 Gaussian quadrature formulas 113
- 3.4 Practical error estimates of elementary quadrature formulas 125
- 3.5 Integration of strongly oscillating functions 130
- 3.6 Increasing the accuracy of integration by equal partition of the interval 134
- 3.7 On statements of optimization problems 139
- 3.8 Optimal quadratures on classes of functions with one derivative 144
- 3.9 Optimization of spacing in a quadrature formula ] 151
- 3.10 Examples of optimization of spacing 157
- 3.11 Principal error term 163
- 3.12 Euler and Gregory formulas 168
- 3.13 Runge's rule for a practical error estimate 171
- 3.14 Romberg's formulas 178
- 3.15 Experiments and their discussion 182
- 3.16 Evaluating integrals in the irregular case 189
- 3.17 Principles of constructing standard programs with automatic choice of step 197
- 3.18 Standard programs of numerical integration 205

4 *Approximation of functions and allied problems* 214

- 4.1 Best approximations in a normed linear space 214
- 4.2 Best approximation in Hilbert space and problems involved in its practical construction 216
- 4.3 Discrete Fourier transform 222
- 4.4 Fast Fourier transform 227
- 4.5 Best uniform approximation 230
- 4.6 Examples of best uniform approximation 233
- 4.7 An iteration method for constructing a polynomial of best uniform approximation 240
- 4.8 On polynomial notation 248
- 4.9 On ways of computing elementary functions 255
- 4.10 On the rate of approximating functions of different classes 260
- 4.11 Interpolation and spline approximation 264
- 4.12 Entropy and  $\epsilon$ -entropy 270

5 *Multidimensional problems* 278

- 5.1 The method of undetermined coefficients 279
- 5.2 The method of least squares 280
- 5.3 The method of regularization 282
- 5.4 An example of regularization 283
- 5.5 Reducing multidimensional problems to one-dimensional ones 290
- 5.6 Error estimation of numerical integration on a uniform grid 298
- 5.7 Lower error estimate in numerical integration 301
- 5.8 On optimizing the error estimate on broader classes of integration methods 304
- 5.9 The Monte Carlo method 309

- 5.10 Justification for using nondeterministic methods in problem solving (a discussion) 314
- 5.11 Accelerating convergence of the Monte Carlo method 316
- 5.12 High-precision quadrature formulas with random points 320
- 5.13 On choosing a method of solving a problem 326

## *Part II*

### PROBLEMS OF ALGEBRA AND OPTIMIZATION 333

- 6 *Numerical methods of algebra* 335
  - 6.1 Methods of elimination 336
  - 6.2 The Method of orthogonalization 344
  - 6.3 The method of simple iteration 347
  - 6.4 Investigation of an actual iteration process 352
  - 6.5 The spectrum of a family of matrices 356
  - 6.6 The  $\delta^2$ -process for a practical estimate of error and acceleration of convergence 361
  - 6.7 Optimizing the rate of convergence of iteration processes 365
  - 6.8 Seidel's method 375
  - 6.9 The method of steepest descent 382
  - 6.10 The method of conjugate gradients 386
  - 6.11 The Monte Carlo method for solving systems of linear equations 392
  - 6.12 Iteration methods involving the use of spectrally equivalent operators 400
  - 6.13 The error of an approximate solution of a system of equations and the condition of the matrix. Regularization 403
  - 6.14 The eigenvalue problem 409
  - 6.15 Solving the complete eigenvalue problem for a symmetric matrix by the rotation method 415
- 7 *Solving systems of nonlinear equations and optimization problems* 420
  - 7.1 The method of simple iteration and allied questions 421
  - 7.2 Newton's method of solving nonlinear equations 426
  - 7.3 Other methods of solving a single equation 431
  - 7.4 Descent methods 436
  - 7.5 Other methods of reducing multidimensional problems to lower-dimensional problems 442
  - 7.6 Solving stationary problems by stabilization 446
  - 7.7 What needs to be optimized? 453
  - 7.8 How do we optimize it? 458

## *Part III*

### NUMERICAL METHODS OF SOLVING ORDINARY DIFFERENTIAL EQUATIONS 465

- 8 *Numerical methods of solving the initial-value problems* 467

- 8.1 Expanding a solution in a Taylor series 468
- 8.2 The Runge-Kutta methods 470
- 8.3 Methods involving error checks at every step 479
- 8.4 Error estimates in one-step methods 481
- 8.5 Finite-difference methods 486
- 8.6 The method of undetermined coefficients 491
- 8.7 Studying the properties of difference methods on model problems 496
- 8.8 Error estimates in difference methods 504
- 8.9 The principal error term 509
- 8.10 Studying the properties of finite-difference methods on more exact models 514
- 8.11 Integrating systems of equations 524
- 8.12 Some general questions 533
- 8.13 Formulas for numerical integration of second-order equations 542
- 8.14 The error estimate of a numerical solution of the initial-value problem for a second-order equation 545
- 8.15 Two-sided methods 551

9 *Numerical methods of solving boundary-value problems for ordinary differential equations* 559

- 9.1 Elementary methods of solving a boundary-value problem for a second-order equation 559
- 9.2 Green's function of a grid boundary-value problem 566
- 9.3 Solving an elementary boundary-value grid problem 571
- 9.4 Closures of computational algorithms 581
- 9.5 Statements of boundary-value problems for linear systems of the first order (a discussion) 589
- 9.6 Solution algorithms of boundary-value problems for systems of first-order equations 595
- 9.7 Methods of differential orthogonal sweeping 601
- 9.8 Nonlinear boundary-value problems 608
- 9.9 Special-type approximations 619
- 9.10 Difference methods for finding eigenvalues 627
- 9.11 Optimization in spacing integration points 631
- 9.12 The effect of a computational error as dependent on the mode of representing the difference equation 638
- 9.13 Estimating the computational error when solving boundary-value problems by the sweep method 643

References 648

Index 655

