Contents

A The Heuristic **1**

A1 The M/M/1 queue. 1
A2 Mosaic processes on $\boldsymbol{R}^2$. 2
A3 Mosaic processes on other spaces. 5
A4 The heuristic. 5
A5 Estimating clump sizes. 7
A6 The harmonic mean formula. 8
A7 Conditioning on semi-local maxima. 10
A8 The renewal-sojourn method. 11
A9 The ergodic-exit method. 12
A10 Limit assertions. 14
A11–A21 Commentary 15

B Markov Chain Hitting Times **23**

B1 Introduction. 23
B2 The heuristic for Markov hitting times. 24
B3 Example: Basic single server queue. 25
B4 Example: Birth-and-death processes. 25
B5 Example: Patterns in coin-tossing. 26
B6 Example: Card-shuffling. 27
B7 Example: Random walk on $\boldsymbol{Z}^d$ mod N. 28
B8 Example: Random trapping on $\boldsymbol{Z}^d$. 28
B9 Example: Two M/M/1 queues in series. 28
B10 Example: Large density of heads in coin-tossing. 29
B11 Counter-example. 30
B12 Hitting small subsets. 30
B13 Example: Patterns in coin-tossing, continued. 31
B14 Example: Runs in biased die-throwing. 31
B15 Example: Random walk on $\boldsymbol{Z}^d$ mod N, continued. 32
B16 Hitting sizable subsets. 32
B17 The ergodic-exit form of the heuristic for Markov hitting times. 32
B18 Example: A simple reliability model. 33
B19 Example: Timesharing computer. 33

B20 Example: Two M/M/1 queues in series. 35
B21 Another queueing example. 37
B22 Example: Random regular graphs. 38
B23–B32 Commentary 38

C Extremes of Stationary Processes **44**
C1 Classical i.i.d. extreme value theory. 44
C2 Examples of maxima of i.i.d. sequences. 45
C3 The point process formulation. 47
C4 The heuristic for dependent extrema. 48
C5 Autoregressive and moving average sequences. 48
C6 Example: Exponential tails. 49
C7 Approximate independence of tail values. 50
C8 Example: Superexponential tails. 50
C9 Example: Polynomial tails. 50
C10 The heuristic for dependent extrema (continued). 51
C11 Additive Markov processes on $[0, \infty)$. 52
C12 Continuous time processes: the smooth case. 54
C13 Example: System response to external shocks. 55
C14 Example: Uniform distribution of the Poisson process. . . . 56
C15 Drift-jump processes. 57
C16 Example: Positive, additive processes. 58
C17 Example: Signed additive processes. 58
C18 Positive, general drift processes. 58
C19 Autoregressive symmetric stable process. 59
C20 Example: The I5 problem. 60
C21 Approximations for the normal distribution. 62
C22 Gaussian processes. 63
C23 The heuristic for smooth Gaussian processes. 64
C24 Monotonicity convention. 64
C25 High-level behavior of smooth Gaussian processes. 65
C26 Conditioning on semi-local maxima. 66
C27 Variations on a theme. 67
C28 Example: Smooth $\mathcal{X}^2$ processes. 68
C29–C40 Commentary 69

D Extremes of Locally Brownian Processes **72**
D1 Brownian motion. 72
D2 The heuristic for locally Brownian processes. 73
D3 One-dimensional diffusions. 74
D4 First hitting times for positive-recurrent diffusions. 76
D5 Example: Gamma diffusion. 77
D6 Example: Reflecting Brownian motion. 78
D7 Example: Diffusions under a potential. 78
D8 Example: State-dependent M/M/1 queue. 79

D9 Example: The Ornstein-Uhlenbeck process. 80
D10 Gaussian processes. 81
D11 Example: System response to external shocks. 82
D12 Example: Maximum of self-normalized Brownian bridge. . . 82
D13 Boundary-crossing. 83
D14 Example: Boundary-crossing for reflecting Brownian motion. 84
D15 Example: Brownian LIL. 85
D16 Maxima and boundary-crossing for general Gaussian processes. 86
D17 Example: Maximum of Brownian bridge. 86
D18 Maxima of non-stationary Gaussian processes. 87
D19 Example: Maximum of Brownian Bridge with drift. 88
D20 Example: Brownian motion and quadratic boundary. 88
D21 Example: Ornstein-Uhlenbeck quadratic boundary. 89
D22 Semi-local maxima for the Ornstein-Uhlenbeck process. . . 90
D23 Example: A storage/queuing process. 91
D24 Approximation by unstable Ornstein-Uhlenbeck process. . . 93
D25 Example: Escape from a potential well. 93
D26 Example: Diffusion in random environment. 94
D27 Interpolating between Gaussian processes. 95
D28 Example: Smoothed Ornstein-Uhlenbeck. 95
D29 Boundary-crossing revisited. 97
D30 Tangent approximation for Brownian boundary-crossing. . . 99
D31–D42 Commentary 100

E Simple Combinatorics **106**
E1 Introduction. 106
E2 Poissonization. 107
E3 Example: The birthday problem. 108
E4 Example: K-matches. 109
E5 Example: Unequal probabilities. 109
E6 Example: Marsaglia random number test. 110
E7 Several types of coincidence. 110
E8 Example: Similar bridge hands. 111
E9 Example: Matching K-sets. 112
E10 Example: Nearby pairs. 112
E11 Example: Basic coupon-collectors problem. 113
E12 Example: Time until most boxes have at least one ball. . . . 113
E13 Example: Time until all boxes have at least $(K+1)$ balls. . 114
E14 Example: Unequal probabilities. 114
E15 Abstract versions of CCP. 114
E16 Example: Counting regular graphs. 115
E17–E22 Commentary 116

F Combinatorics for Processes 118
F1 Birthday problem for Markov chains. 118
F2 Example: Simple random walk on Z^K. 119
F3 Example: Random walks with large step. 119
F4 Example: Simple random walk on the K-cube. 120
F5 Example: Another card shuffle. 120
F6 Matching problems. 120
F7 Matching blocks. 122
F8 Example: Matching blocks: the i.i.d. case. 122
F9 Example: Matching blocks: the Markov case. 123
F10 Birthday problem for blocks. 124
F11 Covering problems. 124
F12 Covering problems for random walks. 125
F13 Example: Random walk on Z^d modulo N. 126
F14 Example: Simple random walk on the K-cube. 126
F15 Covering problem for i.i.d. blocks. 127
F16 Example: Dispersal of many walks. 127
F17 Example: M/M/∞ combinatorics. 128
F18–F21 Commentary 129

G Exponential Combinatorial Extrema 131
G1 Introduction. 131
G2 Example: Cliques in random graphs. 132
G3 Example: Covering problem on the K-cube. 133
G4 Example: Optimum partitioning of numbers. 134
G5 Exponential sums. 135
G6 Example: First-passage percolation on the binary tree. . . . 136
G7 Example: Percolation on the K-cube. 138
G8 Example: Bayesian binary strings. 139
G9 Example: Common cycle partitions in random permutations. 140
G10 Conditioning on maxima. 141
G11 Example: Common subsequences in fair coin-tossing. 141
G12 Example: Anticliques in sparse random graphs. 142
G13 The harmonic mean formula. 143
G14 Example: Partitioning sparse random graphs. 143
G15 Tree-indexed processes. 145
G16 Example: An additive process. 145
G17 Example: An extremal process. 146
G18–G22 Commentary 147

H Stochastic Geometry 149
H1 Example: Holes and clusters in random scatter. 149
H2 The Poisson line process. 151
H3 A clump size calculation. 152
H4 Example: Empty squares in random scatter. 154

H5 Example: Empty rectangles in random scatter. 156
H6 Example: Overlapping random squares. 157
H7 Example: Covering K times. 159
H8 Example: Several types of particle. 159
H9 Example: Non-uniform distributions. 160
H10 Example: Monochrome squares on colored lattice. 161
H11 Example: Caps and great circles. 161
H12 Example: Covering the line with intervals of random length. 162
H13 Example: Clusters in 1-dimensional Poisson processes. . . . 164
H14–H21 Commentary 165

I Multi-Dimensional Diffusions **167**
I1 Background. 167
I2 The heuristic. 169
I3 Potential function. 169
I4 Reversible diffusions. 169
I5 Ornstein-Uhlenbeck processes. 170
I6 Brownian motion on surface of sphere. 170
I7 Local approximations. 170
I8 Example: Hitting times to small balls. 171
I9 Example: Near misses of moving particles. 171
I10 Example: A simple aggregation-disaggregation model. . . . 173
I11 Example: Extremes for diffusions controlled by potentials. . 173
I12 Example: Escape from potential wells. 176
I13 Physical diffusions: Kramers' equation. 177
I14 Example: Extreme values. 178
I15 Example: Escape from potential well. 180
I16 Example: Lower boundaries for transient Brownian motion. 182
I17 Example: Brownian motion on surface of sphere. 183
I18 Rice's formula for conditionally locally Brownian processes. 185
I19 Example: Rough $\mathcal{X}^2$ processes. 186
I20–I29 Commentary 186

J Random Fields **190**
J1 Spatial processes. 190
J2 In analysis of 1-parameter processes. 190
J3 Gaussian fields and white noise. 190
J4 Analogues of the Kolmogorov-Smirnov test. 191
J5 The heuristic. 192
J6 Discrete processes. 192
J7 Example: Smooth Gaussian fields. 193
J8 Example: 2-dimensional shot noise. 195
J9 Uncorrelated orthogonal increments Gaussian processes. . . 195
J10 Example: Product Ornstein-Uhlenbeck processes. 196
J11 An artificial example. 197

J12 Maxima of μ-Brownian sheets. 198
J13 1-parameter Brownian bridge. 198
J14 Example: Stationary × Brownian bridge processes. 200
J15 Example: Range of Brownian bridge. 201
J16 Example: Multidimensional Kolmogorov-Smirnov. 202
J17 Example: Rectangle-indexed sheets. 204
J18 Isotropic Gaussian processes. 205
J19 Slepian's inequality. 206
J20 Bounds from the harmonic mean. 207
J21 Example: Hemispherical caps. 208
J22 Example: Half-plane indexed sheets. 209
J23 The power formula. 212
J24 Self-normalized Gaussian fields. 212
J25 Example: Self-normalized Brownian motion increments. . . 212
J26 Example: Self-normalized Brownian bridge increments. . . . 213
J27 Example: Upturns in Brownian bridge with drift. 214
J28 Example: 2-parameter LIL. 215
J29–J37 Commentary 216

K Brownian Motion: Local Distributions **220**
K1 Modulus of continuity. 220
K2 Example: The Chung-Erdos-Sirao test. 221
K3 Example: The asymptotic distribution of W. 221
K4 Example: Spikes in Brownian motion. 224
K5 Example: Small increments. 225
K6 Example: Integral tests for small increments. 228
K7 Example: Local maxima and points of increase. 230
K8 Example: Self-intersections of d-dimensional Brownian motion. 233
K9–K17 Commentary 234

L Miscellaneous Examples **237**
L1 Example: Meetings of empirical distribution functions. . . . 237
L2 Example: Maximal k-spacing. 238
L3 Example: Increasing runs in i.i.d. sequences. 239
L4 Example: Growing arcs on a circle. 239
L5 Example: The LIL for symmetric stable processes. 241
L6 Example: Min-max of process. 242
L7 2-dimensional random walk 243
L8 Example: Random walk on $\boldsymbol{Z}^2$ modulo N. 244
L9 Example: Covering problems for 2-dimensional walks. . . . 244

M The Eigenvalue Method **246**
M1 Introduction. 246
M2 The asymptotic geometric clump principle. 247
M3 Example: Runs in subsets of Markov chains. 248
M4 Example: Coincident Markov chains. 248
M5 Example: Alternating runs in i.i.d. sequences. 248
M6 Example: Longest busy period in M/G/1 queue. 250
M7 Example: Longest interior sojourn of a diffusion. 250
M8 Example: Boundary crossing for diffusions. 251

Postscript **252**

Bibliography **253**

Index **267**