

CONTENTS

Chapter I Operations with Vectors

1. The vector notation	1
2. Addition of vectors	2
3. Multiplication by scalars.	3
4. Representation of a vector by means of linearly independent vectors	3
5. Scalar product	3
6. Vector product	5
7. Scalar triple product	6
8. Invariance under orthogonal transformations	7
9. Vector calculus	9

Chapter II Plane Curves

1. Introduction	12
2. Regular curves	12
3. Change of parameters	14
4. Invariance under changes of parameter	16
5. Tangent lines and tangent vectors of a curve	16
6. Orientation of a curve	18
7. Length of a curve	19
8. Arc length as an invariant	20
9. Curvature of plane curves	21
10. The normal vector and the sign of κ	23
11. Formulas for κ	26
12. Existence of a plane curve for given curvature κ	27
13. Frenet equations for plane curves	28
14. Evolute and involute of a plane curve	29
15. Envelopes of families of curves	31
16. The Jordan theorem as a problem in differential geometry in the large	34
17. Additional properties of Jordan curves	41
18. The total curvature of a regular Jordan curve	45
19. Simple closed curves with $\kappa \neq 0$ as boundaries of convex point sets	46
20. Four vertex theorem	48

Chapter III Space Curves

1. Regular curves	53
2. Length of a curve	54
3. Curvature of space curves	54
4. Principal normal and osculating plane	55
5. Binormal vector	57
6. Torsion τ of a space curve	57
7. The Frenet equations for space curves	58
8. Rigid body motions and the rotation vector	58
9. The Darboux vector	62
10. Formulas for κ and τ	63
11. The sign of τ	63
12. Canonical representation of a curve	64
13. Existence and uniqueness of a space curve for given $\kappa(s)$, $\tau(s)$	65
14. What about $\kappa = 0$?	67
15. Another way to define space curves	68
16. Some special curves	70

Chapter IV The Basic Elements of Surface Theory

1. Regular surfaces in Euclidean space	74
2. Change of parameters	75
3. Curvilinear coordinate curves on a surface	76
4. Tangent plane and normal vector	77
5. Length of curves and first fundamental form	78
6. Invariance of the first fundamental form	78
7. Angle measurement on surfaces	80
8. Area of a surface	82
9. A few examples	83
10. Second fundamental form of a surface	85
11. Osculating paraboloid	86
12. Curvature of curves on a surface	88
13. Principal directions and principal curvatures	91
14. Mean curvature H and Gaussian curvature K	92
15. Another definition of the Gaussian curvature K	93
16. Lines of curvature	95
17. Third fundamental form	98
18. Characterization of the sphere as a locus of umbilical points	99
19. Asymptotic lines	100
20. Torsion of asymptotic lines	100
21. Introduction of special parameter curves	101
22. Asymptotic lines and lines of curvature as parameter curves	103
23. Embedding a given arc in a system of parameter curves	103
24. Analogues of polar coordinates on a surface	104

Chapter V Some Special Surfaces

1. Surfaces of revolution	109
2. Developable surfaces in the small made up of parabolic points	114
3. Edge of regression of a developable	118
4. Why the name developable?	120
5. Developable surfaces in the large	121
6. Developables as envelopes of planes	129

Chapter VI The Partial Differential Equations of Surface Theory

1. Introduction	133
2. The Gauss equations	134
3. The Christoffel symbols evaluated	135
4. The Weingarten equations	136
5. Some observations about the partial differential equations	136
6. Uniqueness of a surface for given g_{ik} and L_{ik}	138
7. The <i>theorema egregium</i> of Gauss	139
8. How Gauss may have hit upon his theorem	141
9. Compatibility conditions in general	143
10. Codazzi-Mainardi equations	144
11. The Gauss <i>theorema egregium</i> again	144
12. Existence of a surface with given g_{ik} and L_{ik}	146
13. An application of the general theory to a problem in the large	148

Chapter VII Inner Differential Geometry in the Small from the Extrinsic Point of View

1. Introduction. Motivations for the basic concepts	151
2. Approximate local parallelism of vectors in a surface	155
3. Parallel transport of vectors along curves in the sense of Levi-Civita	157
4. Properties of parallel fields of vectors along curves	160
5. Parallel transport is independent of the path only for surfaces having $K \equiv 0$	162
6. The curvature of curves in a surface: the geodetic curvature	163
7. First definition of geodetic lines: lines with $\kappa_g = 0$	165
8. Geodetic lines as candidates for shortest arcs	167
9. Straight lines as shortest arcs in the Euclidean plane	168
10. A general necessary condition for a shortest arc.	171
11. Geodesics in the small and geodetic coordinate systems	174
12. Geodesics as shortest arcs in the small	178
13. Further developments relating to geodetic coordinate systems	179
14. Surfaces of constant Gaussian curvature	183
15. Parallel fields from a new point of view	184
16. Models provided by differential geometry for non-Euclidean geometries	185
17. Parallel transport of a vector around a simple closed curve	191

18. Derivation of the Gauss-Bonnet formula	195
19. Consequences of the Gauss-Bonnet formula	196
20. Tchebychef nets	198

Chapter VIII Differential Geometry in the Large

1. Introduction. Definition of n -dimensional manifolds.	203
2. Definition of a Riemannian manifold	206
3. Facts from topology relating to two-dimensional manifolds.	211
4. Surfaces in three-dimensional space	217
5. Abstract surfaces as metric spaces	218
6. Complete surfaces and the existence of shortest arcs	220
7. Angle comparison theorems for geodetic triangles	226
8. Geodetically convex domains	231
9. The Gauss-Bonnet formula applied to closed surfaces	237
10. Vector fields on surfaces and their singularities	239
11. Poincaré's theorem on the sum of the indices on closed surfaces	244
12. Conjugate points. Jacobi's conditions for shortest arcs	247
13. The theorem of Bonnet-Hopf-Rinow	254
14. Synge's theorem in two dimensions	255
15. Covering surfaces of complete surfaces having $K \leq 0$	259
16. Hilbert's theorem on surfaces in E^3 with $K \equiv -1$	265
17. The form of complete surfaces of positive curvature in three-dimensional space	272

Chapter IX Intrinsic Differential Geometry of Manifolds. Relativity

1. Introduction	282
Part I. Tensor Calculus in Affine and Euclidean Spaces	
2. Affine geometry in curvilinear coordinates.	284
3. Tensor calculus in Euclidean spaces	287
4. Tensor calculus in mechanics and physics	292
Part II. Tensor Calculus and Differential Geometry in General Manifolds	
5. Tensors in a Riemannian space	294
6. Basic concepts of Riemannian geometry	296
7. Parallel displacement. Necessary condition for Euclidean metrics	300
8. Normal coordinates. Curvature in Riemannian geometry	307
9. Geodetic lines as shortest connections in the small	310
10. Geodetic lines as shortest connections in the large	311
Part III. Theory of Relativity	
11. Special theory of relativity	318
12. Relativistic dynamics	323
13. The general theory of relativity	326

Chapter X	The Wedge Product and the Exterior Derivative of Differential Forms, with Applications to Surface Theory	
1.	Definitions	335
2.	Vector differential forms and surface theory	342
3.	Scalar and vector products of vector forms on surfaces and their exterior derivatives	349
4.	Some formulas for closed surfaces. Characterizations of the sphere	351
5.	Minimal surfaces	356
6.	Uniqueness theorems for closed convex surfaces	358
Appendix A	Tensor Algebra in Affine, Euclidean, and Minkowski Spaces	
1.	Introduction	371
2.	Geometry in an affine space	371
3.	Tensor algebra in centered affine spaces	375
4.	Effect of a change of basis	378
5.	Definition of tensors	380
6.	Tensor algebra in Euclidean spaces	385
Appendix B	Differential Equations	
1.	Theorems on ordinary differential equations	388
2.	Overdetermined systems of linear partial differential equations	392
Bibliography	396
Index	401