Contents

Preface ix

I Sobolev spaces **1**

§1 Notation, basic properties, distributions **1**

1.1 Notation 1

1.2 Partition of unity 4

1.3 Regularisation of functions 8

1.4 Distributions 10

1.5 The support of a distribution 12

1.6 Differentiation and multiplication 14

1.7 Distributions with compact support 18

1.8 Convolution 20

1.9 The Fourier transformation 25

§2 Geometric assumptions for the domain Ω **35**

2.1 Segment and cone properties 36

2.2 The $N^{k,\kappa}$-property of Ω 38

2.3 (k,κ)-diffeomorphisms and (k,κ)-smooth Ωs 47

2.4 Normal transformations 52

2.5 Differentiable manifolds 58

§3 Definitions and density properties for the Sobolev–Slobodeckii spaces $W_2^l(\Omega)$ **61**

3.1 Definition of the Sobolev–Slobodeckii spaces $W_2^l(\Omega)$ 61

3.2 Density properties 64

§4 The transformation theorem and Sobolev spaces on differentiable manifolds **74**

4.1 The transformation theorem 75

4.2 Sobolev spaces on differentiable manifolds, and on the frontier $\partial\Omega$ of a (k,κ)-smooth region 87

§5 Definition of Sobolev spaces by the Fourier transformation and extension theorems **90**

5.1 Sobolev spaces and the Fourier transformation 91

5.2 Extension theorems 95

§6 Continuous embeddings and Sobolev's lemma **105**

§7 Compact embeddings 112
§8 The trace operator 120
§9 Weak sequential compactness and approximation of derivatives by difference quotients 133

II Elliptic differential operators 139
§10 Linear differential operators 139
§11 The Lopatinskiĭ–Šapiro condition and examples 148
11.1 The Lopatinskiĭ–Šapiro condition 148
11.2 Examples 157
§12 Fredholm operators 165
12.1 The Riesz–Schauder spectral theorem (compact operators) 165
12.2 Fredholm operators 168
12.3 A priori *estimates, the Weyl lemma and smoothable operators* 180
§13 The main theorem and some theorems on the index of elliptic boundary value problems 186
13.1 The main theorems for elliptic boundary value problems 186
13.2 The index and spectrum of elliptic boundary value problems 209
§14 Green's formulae 213
14.1 Normal boundary value operators and Dirichelet systems 214
14.2 The first Green formula 219
14.3 Adjoint boundary value operators and boundary value spaces 222
14.4 The second Green formula 231
14.5 The antidual operator L′ and the adjoint boundary value problem 235
§15 The adjoint boundary value problem and the connection with the image space of the original operator 239
§16 Examples 252

III Strongly elliptic differential operators and the method of variations 261
§17 Gelfand triples, the Lax–Milgram theorem, *V*-elliptic and *V*-coercive operators 261
17.1 Gelfand triples 261
17.2 Representations for functionals on Sobolev spaces 268
17.3 The Lax–Milgram theorem 271
17.4 V-elliptic and V-coercive forms, solution theorems 273
17.5 The Green operator 275
17.6 The concepts V-elliptic and V-coercive for differential operators 279
§18 Agmon's condition 280
§19 Agmon's theorem: conditions for the *V*-coercion of strongly elliptic differential operators 290
19.1 The theorems of Gårding and Agmon 290
19.2 Examples, including the Dirichlet problem for strongly elliptic differential operators 302

§20 Regularity of the solutions of strongly elliptic equations 307
§21 The solution theorem for strongly elliptic equations and examples 336
§22 The Schauder fixed point theorem and a non-linear problem 361
§23 Elliptic boundary value problems for unbounded regions 370

IV Parabolic differential operators 376

§24 The Bochner integral 376
24.1 Pettis' theorem 376
24.2 The Bochner integral 384
§25 Distributions with values in a Hilbert space H and the space $W(0, T)$ 390
§26 The existence and uniqueness of the solution of a parabolic differential equation 395
§27 The regularity of solutions of the parabolic differential equation 403
27.1 An abstract regularity theorem 404
27.2 Differentiability with respect to t 411
27.3 Differentiability with respect to x, respectively t 414
§28 Examples 423

V Hyperbolic differential operators 434

§29 Existence and uniqueness of the solution 434
§30 Regularity of the solutions of the hyperbolic differential equation 442
30.1 An abstract regularity theorem 442
30.2 Differentiability with respect to t 445
30.3 Differentiability with respect to x 447
§31 Examples 452

VI Difference processes for the calculation of the solution of the partial differential equation 462

§32 Functional analytic concepts for difference processes 462
§33 Difference processes for elliptic differential equations and for the wave equation 481
33.1 Some important inequalities 481
33.2 Construction of a difference process for the Dirichlet problem 484
33.3 A difference process for the wave equation in several space variables 488
§34 Evolution equations 496
34.1 The time-independent case 498
34.2 The time-dependent case 503
34.3 Stability behaviour of the perturbed process 505
34.4 Several step processes 507

References 511
Function and distribution spaces 515
Index 516