

Table of Contents

Chapter 1	General Discretization Methods	1
1.1.	Basic Definitions	1
1.1.1	Discretization Methods	1
1.1.2	Consistency	5
1.1.3	Convergence	7
1.1.4	Stability	9
1.2	Results Concerning Stability	11
1.2.1	Existence of the Solution of the Discretization	11
1.2.2	The Basic Convergence Theorem	13
1.2.3	Linearization	15
1.2.4	Stability of Neighboring Discretizations	18
1.3	Asymptotic Expansions of the Discretization Errors	21
1.3.1	Asymptotic Expansion of the Local Discretization Error	21
1.3.2	Asymptotic Expansion of the Global Discretization Error	25
1.3.3	Asymptotic Expansions in Even Powers of n	29
1.3.4	The Principal Error Terms	30
1.4	Applications of Asymptotic Expansions	33
1.4.1	Richardson Extrapolation	33
1.4.2	Linear Extrapolation	37
1.4.3	Rational Extrapolation	42
1.4.4	Difference Correction	44
1.5	Error Analysis	49
1.5.1	Computing Error	49
1.5.2	Error Estimates	51
1.5.3	Strong Stability	54
1.5.4	Richardson-extrapolation and Error Estimation	56
1.5.5	Statistical Analysis of Round-off Errors	58
1.6	Practical Aspects	61

Chapter 2 Forward Step Methods	63
2.1 Preliminaries	63
2.1.1 Initial Value Problems for Ordinary Differential Equations	63
2.1.2 Grids	65
2.1.3 Characterization of Forward Step Methods	67
2.1.4 Restricting the Interval	69
2.1.5 Notation	72
2.2 The Meaning of Consistency, Convergence, and Stability with Forward Step Methods	74
2.2.1 Our Choice of Norms in E_n and E_n^0	74
2.2.2 Other Definitions of Consistency and Convergence	76
2.2.3 Other Definitions of Stability	79
2.2.4 Spijker's Norm for E_n^0	81
2.2.5 Stability of Neighboring Discretizations	84
2.3 Strong Stability of f. s. m.	87
2.3.1 Perturbation of IVP 1	87
2.3.2 Discretizations of $\{\text{IVP 1}\}_\tau$	92
2.3.3 Exponential Stability for Difference Equations on $[0, \infty)$	94
2.3.4 Exponential Stability of Neighboring Discretizations	98
2.3.5 Strong Exponential Stability	101
2.3.6 Stability Regions	103
2.3.7 Stiff Systems of Differential Equations	104
Chapter 3 Runge-Kutta Methods	107
3.1 RK-procedures	107
3.1.1 Characterization	107
3.1.2 Local Solution and Increment Function	110
3.1.3 Elementary Differentials	111
3.1.4 The Expansion of the Local Solution	115
3.1.5 The Exact Increment Function	117
3.2 The Group of RK-schemes	120
3.2.1 RK-schemes	120
3.2.2 Inverses of RK-schemes	125
3.2.3 Equivalent Generating Matrices	127
3.2.4 Explicit and Implicit RK-schemes	131
3.2.5 Symmetric RK-procedures	134
3.3 RK-methods and Their Orders	135
3.3.1 RK-methods	135
3.3.2 The Order of Consistency	137

3.3.3 Construction of High-order RK-procedures	141
3.3.4 Attainable Order of m -stage RK-procedures	144
3.3.5 Effective Order of RK-schemes	148
3.4 Analysis of the Discretization Error	150
3.4.1 The Principal Error Function	150
3.4.2 Asymptotic Expansion of the Discretization Error	153
3.4.3 The Principal Term of the Global Discretization Error	157
3.4.4 Estimation of the Local Discretization Error	161
3.5 Strong Stability of RK-methods	165
3.5.1 Strong Stability for Sufficiently Large n	165
3.5.2 Strong Stability for Arbitrary n	170
3.5.3 Stability Regions of RK-methods	174
3.5.4 Use of Stability Regions for General $\{IVP 1\}_T$	178
3.5.5 Suggestion for a General Approach	182
Chapter 4 Linear Multistep Methods	185
4.1 Linear k -step Schemes	185
4.1.1 Characterization	185
4.1.2 The Order of Linear k -step Schemes	191
4.1.3 Construction of Linear k -step Schemes of High Order	195
4.2 Uniform Linear k -step Methods	199
4.2.1 Characterization, Consistency	199
4.2.2 Auxiliary Results	203
4.2.3 Stability of Uniform Linear k -step Methods	206
4.2.4 Convergence	210
4.2.5 Highest Obtainable Orders of Convergence	214
4.3 Cyclic Linear k -step Methods	216
4.3.1 Stability of Cyclic Linear k -step Methods	216
4.3.2 The Auxiliary Method	221
4.3.3 Attainable Order of Cyclic Linear Multistep Methods	225
4.4 Asymptotic Expansions	228
4.4.1 The Local Discretization Error	228
4.4.2 Asymptotic Expansion of the Global Discretization Error, Preparations	232
4.4.3 The Case of No Extraneous Essential Zeros	234
4.4.4 The Case of Extraneous Essential Zeros	240
4.5 Further Analysis of the Discretization Error	245
4.5.1 Weak Stability	245
4.5.2 Smoothing	248

4.5.3	Symmetric Linear k -step Schemes	250
4.5.4	Asymptotic Expansions in Powers of h^2	256
4.5.5	Estimation of the Discretization Error	260
4.6	Strong Stability of Linear Multistep Methods	263
4.6.1	Strong Stability for Sufficiently Large n	263
4.6.2	Stability Regions of Linear Multistep Methods	266
4.6.3	Strong Stability for Arbitrary n	270
Chapter 5	Multistage Multistep Methods	272
5.1	General Analysis	272
5.1.1	A General Class of Multistage Multistep Procedures	272
5.1.2	Simple m -stage k -step Methods	275
5.1.3	Stability and Convergence of Simple m -stage k -step Methods	278
5.2	Predictor-corrector Methods	282
5.2.1	Characterization, Subclasses	282
5.2.2	Stability and Order of Predictor-corrector Methods	284
5.2.3	Analysis of the Discretization Error	290
5.2.4	Estimation of the Local Discretization Error	294
5.2.5	Estimation of the Global Discretization Error	297
5.3	Predictor-corrector Methods with Off-step Points	300
5.3.1	Characterization	300
5.3.2	Determination of the Coefficients and Attainable Order	302
5.3.3	Stability of High Order PC-methods with Off-step Points	306
5.4	Cyclic Forward Step Methods	308
5.4.1	Characterization	308
5.4.2	Stability and Error Propagation	311
5.4.3	Primitive m -cyclic k -step Methods	316
5.4.4	General Straight m -cyclic k -step Methods	322
5.5	Strong Stability	324
5.5.1	Characteristic Polynomial, Stability Regions	324
5.5.2	Stability Regions of PC-methods	326
5.5.3	Stability Regions of Cyclic Methods	329
Chapter 6	Other Discretization Methods for IVP 1	332
6.1	Discretization Methods with Derivatives of f	332
6.1.1	Recursive Computation of Higher Derivatives of the Local Solution.	332
6.1.2	Power Series Methods	335

6.1.3	The Perturbation Theory of Groebner-Knapp-Wanner	336
6.1.4	Groebner-Knapp-Wanner Methods	339
6.1.5	Runge-Kutta-Fehlberg Methods	343
6.1.6	Multistep Methods with Higher Derivatives	347
6.2	General Multi-value Methods	349
6.2.1	Nordsieck's Approach	349
6.2.2	Nordsieck Predictor-corrector Methods	354
6.2.3	Equivalence of Generalized Nordsieck Methods	357
6.2.4	Appraisal of Nordsieck Methods	361
6.3	Extrapolation Methods	362
6.3.1	The Structure of an Extrapolation Method	362
6.3.2	Gragg's Method	364
6.3.3	Strong Stability of \mathfrak{M}_G	369
6.3.4	The Gragg-Bulirsch-Stoer Extrapolation Method	372
6.3.5	Extrapolation Methods for Stiff Systems	375
	Bibliography	380
	Subject Index	385

