Contents

Preface *vii*

1. Functionals *1*

1.1. Introduction; Examples of Optimization Problems 1
1.2. Vector Spaces 6
1.3. Functionals 9
1.4. Normed Vector Spaces 18
1.5. Continuous Functionals 23
1.6. Linear Functionals 28

2. A Fundamental Necessary Condition for an Extremum *31*

2.1. Introduction 31
2.2. A Fundamental Necessary Condition for an Extremum 33
2.3. Some Remarks on the Gâteaux Variation 38
2.4. Examples on the Calculation of Gâteaux Variations 41
2.5. An Optimization Problem in Production Planning 48
2.6. Some Remarks on the Fréchet Differential 55

3. *The Euler-Lagrange Necessary Condition for an Extremum with Constraints* 58

3.1. Extremum Problems with a Single Constraint 58
3.2. Weak Continuity of Variations 60
3.3. Statement of the Euler-Lagrange Multiplier Theorem for a Single Constraint 62
3.4. Three Examples, and Some Remarks on the Geometrical Significance of the Multiplier Theorem 64
3.5. Proof of the Euler-Lagrange Multiplier Theorem 72
3.6. The Euler-Lagrange Multiplier Theorem for Many Constraints 77
3.7. An Optimum Consumption Policy with Terminal Savings Constraint During a Period of Inflation 80
3.8. The Meaning of the Euler-Lagrange Multipliers 89
3.9. Chaplygin's Problem, or a Modern Version of Queen Dido's Problem 95
3.10. The John Multiplier Theorem 108

4. *Applications of the Euler-Lagrange Multiplier Theorem in the Calculus of Variations* 113

4.1. Problems with Fixed End Points 114
4.2. John Bernoulli's Brachistochrone Problem, and Brachistochrones Through the Earth 126
4.3. Geodesic Curves 138
4.4. Problems with Variable End Points 149
4.5. How to Design a Thrilling Chute-the-Chute 168
4.6. Functionals Involving Several Unknown Functions 178
4.7. Fermat's Principle in Geometrical Optics 186
4.8. Hamilton's Principle of Stationary Action; an Example on Small Vibrations 195
4.9. The McShane-Blankinship Curtain Rod Problem; Functionals Involving Higher-Order Derivatives 206
4.10. Functionals Involving Several Independent Variables; the Minimal Surface Problem 217
4.11. The Vibrating String 227

5. *Applications of the Euler-Lagrange Multiplier Theorem to Problems with Global Pointwise Inequality Constraints* 233

5.1. Slack Functions and Composite Curves 233
5.2. An Optimum Consumption Policy with Terminal Savings Constraint Without Extreme Hardship 247
5.3. A Problem in Production Planning with Inequality Constraints 261

6. *Applications of the Euler-Lagrange Multiplier Theorem in Elementary Control Theory* **274**

6.1. Introduction 274
6.2. A Rocket Control Problem: Minimum Time 277
6.3. A Rocket Control Problem: Minimum Fuel 283
6.4. A More General Control Problem 288
6.5. A Simple Bang-Bang Problem 297
6.6. Some Remarks on the Maximum Principle and Dynamic Programming 307

7. *The Variational Description of Sturm-Liouville Eigenvalues* **309**

7.1. Introduction to Sturm-Liouville Problems 310
7.2. The Relation Between the Lowest Eigenvalue and the Rayleigh Quotient 314
7.3. The Rayleigh-Ritz Method for the Lowest Eigenvalue 318
7.4. Higher Eigenvalues and the Rayleigh Quotient 323
7.5. The Courant Minimax Principle 327
7.6. Some Implications of the Courant Minimax Principle 331
7.7. Further Extensions of the Theory 335
7.8. Some General Remarks on the Ritz Method of Approximate Minimization 338

8. *Some Remarks on the Use of the Second Variation in Extremum Problems* **343**

8.1. Higher-Order Variations 343
8.2. A Necessary Condition Involving the Second Variation at an Extremum 347
8.3. Sufficient Conditions for a Local Extremum 348

Appendix **352**

A1. The Cauchy and Schwarz Inequalities 352
A2. An Example on Normed Vector Spaces 353
A3. An Integral Inequality 355
A4. A Fundamental Lemma of the Calculus of Variations 355
A5. Du Bois-Reymond's Derivation of the Euler-Lagrange Equation 357
A6. A Useful Result from Calculus 360
A7. The Construction of a Certain Function 362
A8. The Fundamental Lemma for the Case of Several Independent Variables 363
A9. The Kinetic Energy for a Certain Model of an Elastic String 364
A10. The Variation of an Initial Value Problem with Respect to a Parameter 366

Subject Index **369**

Author Index **375**