Contents

	Preface	xii
1	**Saddle function problems**	**1**
1.1	The basic idea	1
	(i) A simple saddle function	1
	(ii) A convex function of one variable	4
	(iii) A general saddle function	6
	(iv) Equivalence problems	10
	Exercises 1.1	12
	(v) Quadratic example	13
	Exercises 1.2	16
	(vi) A saddle quantity	17
1.2	Inequality constraints	21
	(i) A single variable example	21
	Exercises 1.3	27
	(ii) An inequality problem generated by a saddle function	28
	(iii) A graphical illustration of equivalence	31
	Exercises 1.4	34
1.3	Transition to higher dimensions	35
	(i) Notation	35
	(ii) Basic equivalence problems	36
	(iii) Definition of a saddle function in higher dimensions	38
	(iv) Second derivative hypotheses	41
	(v) Quadratic example	43
	(vi) Invariance in saddle definitions	44
	Exercises 1.5	46
1.4	Upper and lower bounds and uniqueness	47
	(i) Introduction	47
	(ii) Upper and lower bounds	47
	(iii) Example	48
	(iv) Uniqueness of solution	50
	(v) Embedding in concave/linear $L[x, u]$	51
	Exercises 1.6	52
1.5	Converse theorems: extremum principles	53
	(i) Introduction	53

(ii) Extremum principles for nonlinear $L[x,u]$ 55
(iii) Maximum principle by embedding 60
(iv) A stationary value problem under constraint 66
Exercises 1.7 70
1.6 Examples of links with other viewpoints 71
(i) Linear programming 71
(ii) Quadratic programming 73
(iii) Decomposition of a nonhomogeneous linear problem 74
(iv) Hypercircle 76
Exercises 1.8 78
1.7 Initial motion problems 79
(i) Mechanical background 79
(ii) Initial motion problem 80
(iii) Generating saddle function 80
(iv) Unilateral constraints 81
(v) Examples 81
(vi) Simultaneous extremum principles 83
(vii) Cavitation 84
1.8 Geometric programming 84
(i) Introduction 84
(ii) Primal problem 85
(iii) Saddle function 85
(iv) Governing conditions 86
(v) Dual problem 86
(vi) Alternative version of the dual problem 87
(vii) Inequality constraints 88
1.9 Allocation problem 90
(i) Introduction 90
(ii) Saddle function and governing conditions 91
(iii) Simultaneous extremum principles 91

2 Duality and Legendre transformations **93**
2.1 Introduction 93
2.2 Legendre transformation 95
(i) Introduction 95
(ii) Duality between a point and a plane 96
(iii) Polar reciprocation 98
(iv) Plane locus of pole dual to point envelope of polar 98
(v) Duality between regular branches 101
(vi) Classification of singularities 103
(vii) Stability of singularities 106
(viii) Bifurcation set 107
Exercises 2.1 108

2.3 Legendre transformations in one active variable, with singularities 108
(i) Inflexion and cusp as dual isolated singularities 108
(ii) Dual distributed and accumulated singularities 112
(iii) Some general theorems 113
Exercises 2.2 114
(iv) Ladder for the cuspoids 115
(v) Half-line dual of a pole at infinity 119
(vi) Response curves and complementary areas 121
(vii) Non-decreasing response characteristics 124
2.4 Legendre transformations in two active variables, with singularities 126
(i) Umbilics, illustrating dual isolated singularities 126
(ii) Accumulated duals of nonplanar distributed singularities 130
Exercises 2.3 131
2.5 Closed chain of Legendre transformations 132
(i) Indicial notation 132
(ii) Inner product space notation 134
(iii) Convex and saddle-shaped branches 135
Exercises 2.4 144
2.6 Examples of quartets of Legendre transformations 145
(i) Introductory examples 145
(ii) Strain energy and complementary energy density 146
(iii) Incremental elastic/plastic constitutive equations 147
Exercises 2.5 153
(iv) Other physical examples of Legendre transformations 156
(v) Stable singularities in a constrained plane mapping 159
2.7 The structure of maximum and minimum principles 159
(i) Bounds and Legendre transformations 159
(ii) Bifurcation theory 161
Exercises 2.6 164
(iii) Generalized Hamiltonian and Lagrangian aspects 166
(iv) Supplementary constraints 168
Exercises 2.7 174
2.8 Network theory 175
(i) Introduction 175
(ii) A simple electrical network 176
(iii) Node-branch incidence matrix 177
(iv) Electrical branch characteristics 178
(v) Saddle function and governing equations 179
(vi) Bounds and extremum principles 180
(vii) Equivalent underdetermined systems 182

(viii) Loop-branch formulation 184
(ix) Branch characteristics with inequalities 186

3 Upper and lower bounds via saddle functionals **188**
3.1 Introduction 188
3.2 Inner product spaces 189
(i) Linear spaces 189
(ii) Inner product spaces 190
3.3 Linear operators and adjointness 193
(i) Operators 193
(ii) Adjointness of linear operators 195
(iii) Examples of adjoint operators 196
*(iv) The operator T*T* 203
Exercises 3.1 207
3.4 Gradients of functionals 208
(i) Derivatives of general operators 209
(ii) Gradients of functionals 211
(iii) Partial gradients of functionals 213
3.5 Saddle functional 216
(i) Saddle quantity 216
(ii) Definition of a saddle functional 216
3.6 Upper and lower bounds 218
(i) Introduction 218
(ii) A central theorem in infinite dimensions 219
(iii) Quadratic generating functional 220
3.7 An ordinary differential equation 221
(i) Introduction 221
(ii) Intermediate variable 222
(iii) Inner product spaces and adjoint operators 222
(iv) Hamiltonian functional 223
(v) Generating functional 224
(vi) Identification of A and B 224
(vii) Stationary principle 225
(viii) Saddle functional $L[\tilde{x}, \tilde{u}]$ 225
(ix) Upper and lower bounds 226
(x) Associated inequality problems 226
(xi) Solution of individual constraints 227
(xii) Evaluation of simultaneous bounds 229
(xiii) Specific example 230
(xiv) The fundamental lemma of the calculus of variations 231
(xv) Complementary stationary principles 233
(xvi) Weighting function with isolated zeros 234
Exercises 3.2 235

3.8 Solution of linear constraints 237
(i) Consistency conditions 237
(ii) General formulae for $J[u_\alpha]$ and $K[x_\beta]$ 239
(iii) Separate upper and lower bounds, not simultaneous 240
(iv) Example of a single bound 242
3.9 A procedure for the derivation of bounds 244
(i) Introduction 244
(ii) Steps in the procedure 245
(iii) Hierarchy of smoothness in admissible fields 247
3.10 A catalogue of examples 249
(i) Introduction 249
(ii) Obstacle problem 249
(iii) Euler equation and Hamilton's principle 251
(iv) Föppl–Hencky equation 255
(v) A partial differential equation 257
(vi) A free boundary problem 260
Exercises 3.3 263
3.11 Variational inequalities 271
(i) Introduction 271
(ii) A general definition 271
(iii) Examples 273
3.12 Nonnegative operator equations 274
(i) Introduction 274
(ii) Examples 275
(iii) General results 277
(iv) Laplacian problems 279
(v) A comparison of equivalent differential and integral equations 280
(vi) Alternative bounds 283
(vii) Wave scattering at a submerged weir 288

4 Extensions of the general approach **293**
4.1 Introduction 293
4.2 Bounds on linear functionals 293
(i) Introduction 293
(ii) Nonnegative operator problems 295
(iii) Other saddle-generated problems 298
Exercises 4.1 301
(iv) Comparison problems for a cantilever beam 303
(v) Other examples of pointwise bounds 306
(vi) Boundedness hypotheses 315
(vii) Embedding method 317
(viii) Examples 319

Exercises 4.2 332
4.3 Initial value problems 324
(i) Introduction 324
(ii) The role of the adjoint problem 324
(iii) A simple example 326
(iv) Change of variable 328
(v) A self-adjoint representation 331
(vi) A particular change of variable 331
(vii) Example 332
(viii) A general first order system 334
(ix) A second order equation 335
(x) A heat conduction problem 336
(xi) Alternative approaches 338
Exercises 4.3 340
4.4 Diffusion of liquid through a porous medium 341
(i) Governing equations 341
(ii) Adjointness 342
(iii) Governing equations as $T\tilde{x} + \tilde{b} = 0$ 343
(iv) Nonnegative T 343
(v) Generating saddle functional 344
(vi) Upper and lower bounds 344
4.5 Comparison methods 345
(i) The basic idea 345
(ii) Example of the general setting 345
(iii) Easy and hard problems 347
(iv) Difference variables 347
(v) Stationary and saddle properties 348
(vi) Hashin–Shtrikman functional 349

5 Mechanics of solids and fluids **353**
5.1 Introduction 353
5.2 Thermodynamics 353
(i) Thermodynamic potential functions 353
(ii) Second derivatives of thermodynamic potentials 355
(iii) Simple fluids 357
Exercises 5.1 359
5.3 Compressible inviscid flow 360
(i) Thermodynamics 360
(ii) Another Legendre transformation 361
(iii) Constitutive surfaces 362
(iv) Balance of mass, momentum and energy 364
(v) Simultaneous upper and lower bounds for steady irrotational flow 365

(vi) General flows 374
Exercises 5.2 376
5.4 Magnetohydrodynamic pipe flow 380
(i) Governing equations 380
(ii) Decomposition method 380
(iii) Embedding method 382
5.5 Inner product spaces and adjoint operators in continuum mechanics 382
(i) Inner product spaces 382
(ii) Adjointness 384
(iii) Virtual work 385
(iv) Discontinuities of stress and displacement 386
5.6 Classical elasticity 387
(i) Governing equations 387
(ii) Generalized Hamiltonian representation 388
(iii) Generating saddle functional 388
(iv) Simultaneous upper and lower bounds 390
5.7 Slow viscous flow 393
(i) Governing equations 393
(ii) Generalized Hamiltonian representation 394
(iii) Generating saddle functional 395
(iv) Historical remarks and applications 396
Exercises 5.3 397
5.8 Rigid/plastic yield point problem 399
(i) Governing equations 399
(ii) Generalized Hamiltonian representation 400
(iii) Generating saddle functional 401
(iv) Uniqueness of stress 402
(v) Simultaneous upper and lower bounds 403
Exercises 5.4 406
5.9 Finite elasticity 408
(i) Strain measures 408
(ii) Stress measures 410
(iii) Hyperelastic solid 412
(iv) Governing equations 413
(v) Stationary principles 413
Exercises 5.5 418
5.10 Incremental elastic/plastic distortion 418
(i) Governing equations 419
(ii) Strain-rate 420
(iii) Stress-rate 420
(iv) Constitutive rate equations 420
(v) Continuing equilibrium 421
(vi) Adjointness 422

(vii) Generating functional 422
(viii) Upper and lower bounds 425
(ix) Weaker hypotheses 428
(x) Kinematical constraints 428
(xi) Equilibrium constraints 432
(xii) Application of supplementary constraints 435
Exercises 5.6 441

References 444
Index 461