

Contents

PART ONE GENERAL THEORY

Chapter 1 Elliptic Functions

1	Liouville Theorems	5
2	The Weierstrass Function	7
3	The Addition Theorem	12
4	Isomorphism Classes of Elliptic Curves	14
5	Endomorphisms and Automorphisms	19

Chapter 2 Homomorphisms

1	Points of Finite Order	23
2	Isogenies	25
3	The Involution	28

Chapter 3 The Modular Function

1	The Modular Group	29
2	Automorphic Functions of Degree $2k$	32
3	The Modular Function j	39

Chapter 4 Fourier Expansions

1	Expansion for G_k, g_2, g_3, Δ and j	43
2	Expansion for the Weierstrass Function	45
3	Bernoulli Numbers	48

Chapter 5 The Modular Equation

1	Integral Matrices with Positive Determinant	51
2	The Modular Equation	54
3	Relations with Isogenies	58

Chapter 6 Higher Levels

1	Congruence Subgroups	61
2	The Field of Modular Functions Over \mathbf{C}	62
3	The Field of Modular Functions Over \mathbf{Q}	65
4	Subfields of the Modular Function Field	72

Chapter 7 Automorphisms of the Modular Function Field

1	Rational Adeles of GL_2	75
2	Operation of the Rational Adeles on the Modular Function Field	77
3	The Shimura Exact Sequence	83

PART TWO COMPLEX MULTIPLICATION**ELLIPTIC CURVES WITH SINGULAR INVARIANTS****Chapter 8 Results from Algebraic Number Theory**

1	Lattices in Quadratic Fields	89
2	Completions	98
3	The Decomposition Group and Frobenius Automorphism	101
4	Summary of Class Field Theory	107

Chapter 9 Reduction of Elliptic Curves

1	Non-degenerate Reduction, General Case	111
2	Reduction of Endomorphisms	112
3	Coverings of Level N	113
4	Reduction of Differential Forms	117

Chapter 10 Complex Multiplication

1	Generation of Class Fields, Deuring's Approach	123
2	Idelic Formulation for Arbitrary Lattices	129
3	Generation of Class Fields by Singular Values of Modular Functions	132
4	The Frobenius Endomorphism	136
	Appendix A Relation of Kronecker	143

Chapter 11 Shimura's Reciprocity Law

- 1 Relation Between Generic and Special Extensions 149
- 2 Application to Quotients of Modular Forms 153

Chapter 12 The Function $\Delta(\alpha z)/\Delta(z)$

- 1 Behavior Under the Artin Automorphism 161
- 2 Prime Factorization of its Values 163
- 3 Analytic Proof for the Congruence Relation of j 168

Chapter 13 The ℓ -adic and p -adic Representations of Deuring

- 1 The ℓ -adic Spaces 172
- 2 Representations in Characteristic p 174
- 3 Representations and Isogenies 178
- 4 Reduction of the Ring of Endomorphisms 181
- 5 The Deuring Lifting Theorem 184

Chapter 14 Ihara's Theory

- 1 Deuring Representatives 187
- 2 The Generic Situation 190
- 3 Special Situations. 191

PART THREE ELLIPTIC CURVES WITH NON-INTEGRAL INVARIANT**Chapter 15 The Tate Parametrization**

- 1 Elliptic Curves with Non-integral Invariants 197
- 2 Elliptic Curves Over a Complete Local Ring 202

Chapter 16 The Isogeny Theorems

- 1 The Galois p -adic Representations 205
- 2 Results of Kummer Theory 208
- 3 The Local Isogeny Theorems 211
- 4 Supersingular Reduction 213
- 5 The Global Isogeny Theorems 216

Chapter 17 Division Points over Number Fields

- 1 A Theorem of Shafarevič 221
- 2 The Irreducibility Theorem 225

3	The Horizontal Galois Group	226
4	The Vertical Galois Group	229
5	End of the Proof	231

PART FOUR THETA FUNCTIONS AND KRONECKER LIMIT FORMULA

Chapter 18 Product Expansions

1	The Sigma and Zeta Function	239
	Appendix The Skew Symmetric Pairing	243
2	A Normalization and the q -product for the σ Function	246
3	q -expansions Again	248
4	q -product for Δ	249
5	The Eta Function of Dedekind	252
6	Modular Functions of Level 2	254

Chapter 19 The Fundamental Theta Function

1	Basic Properties	259
2	The Siegel Functions	261
3	Special Values of the Siegel Functions	264

Chapter 20 The Kronecker Limit Formulas

1	The Poisson Summation Formula	267
2	Examples	268
3	The Function $K_s(x)$	270
4	The Kronecker First Limit Formula	273
5	The Kronecker Second Limit Formula	276

Chapter 21 The First Limit Formula and L-series

1	Relation with L-series	279
2	The Frobenius Determinant	284
3	Application to the L-series	286

Chapter 22 The Second Limit Formula and L-series

1	Gauss Sums	287
2	An Expression for the L-series	289

Appendix 1 BY J. TATE**Algebraic Formulas in Arbitrary Characteristic**

- | | | |
|---|--|-----|
| 1 | Generalized Weierstrass Form | 299 |
| 2 | Canonical Forms | 301 |

Appendix 2 **The Trace of Frobenius and the Differential of the First Kind**

- | | | |
|---|----------------------------------|-----|
| 1 | The Trace of Frobenius | 307 |
| 2 | Duality | 308 |
| 3 | The Tate Trace | 309 |
| 4 | The Cartier Operator | 311 |
| 5 | The Hasse Invariant | 316 |

Bibliography 321

Index 325