

Contents

Translator's Note	v
Foreword to the First Russian Edition	vii
Foreword to the Second Russian Edition	xi

Chapter I

Definition and Simplest Properties of Generalized Functions	1
1. Test Functions and Generalized Functions	1
1.1. Introductory Remarks	1
1.2. Test Functions	2
1.3. Generalized Functions	3
1.4. Local Properties of Generalized Functions	5
1.5. Addition and Multiplication by a Number and by a Function	7
1.6. Translations, Rotations, and Other Linear Transformations on the Independent Variables	8
1.7. Regularization of Divergent Integrals	10
1.8. Convergence of Generalized Function Sequences	13
1.9. Complex Test Functions and Generalized Functions	15
1.10. Other Test-Function Spaces	16
2. Differentiation and Integration of Generalized Functions	18
2.1. Fundamental Definitions	18
2.2. Examples for the Case of a Single Variable	21
2.3. Examples for the Case of Several Variables	27
2.4. Differentiation as a Continuous Operation	29
2.5. Delta-Convergent Sequences	34
2.6. Differential Equations for Generalized Functions	39
2.7. Differentiation in S	44
3. Regularization of Functions with Algebraic Singularities	45
3.1. Statement of the Problem	45
3.2. The Generalized Functions x_+^{λ} and x_-^{λ}	48
3.3. Even and Odd Combinations of x_+^{λ} and x_-^{λ}	50

3.4. Indefinite Integrals of x_+^λ , x_-^λ , $ x ^\lambda \operatorname{sgn} x$	54
3.5. Normalization of x_+^λ , x_-^λ , $ x ^\lambda \operatorname{sgn} x$	55
3.6. The Generalized Functions $(x + i0)^\lambda$ and $(x - i0)^\lambda$	59
3.7. Canonical Regularization	61
3.8. Regularization of Other Integrals	65
3.9. The Generalized Function r^λ	71
3.10. Plane-Wave Expansion of r^λ	74
3.10. Homogeneous Functions	78
 4. Associate Functions	82
4.1. Definition	82
4.2. Taylor's and Laurent Series for x_+^λ and x_-^λ	84
4.3. Expansion of $ x ^\lambda$ and $ x ^\lambda \operatorname{sgn} x$	89
4.4. The Generalized Functions $(x + i0)^\lambda$ and $(x - i0)^\lambda$	93
4.5. Taylor's Series for $(x + i0)^\lambda$ and $(x - i0)^\lambda$	96
4.6. Expansion of r^λ	98
 5. Convolutions of Generalized Functions	100
5.1. Direct Product of Generalized Functions	100
5.2. Convolutions of Generalized Functions	103
5.3. Newtonian Gravitational Potential and Elementary Solutions of Differential Equations	106
5.4. Poisson's Integral and Elementary Solutions of Cauchy's Problem	109
5.5. Integrals and Derivatives of Higher Orders	115
 6. Elementary Solutions of Differential Equations with Constant Coefficients	122
6.1. Elementary Solutions of Elliptic Equations	122
6.2. Elementary Solutions of Regular Homogeneous Equations	128
6.3. Elementary Solutions of Cauchy's Problem	132
 Appendix 1. Local Properties of Generalized Functions	140
A1.1. Test Functions as Averages of Continuous Functions	141
A1.2. Partition of Unity	142
A1.3. Local Properties of Generalized Functions	144
A1.4. Differentiation as a Local Operation	146

Appendix 2. Generalized Functions Depending on a Parameter	147
A2.1. Continuous Functions	147
A2.2. Differentiable Functions	148
A2.3. Analytic Functions	149
Chapter II	
Fourier Transforms of Generalized Functions	153
1. Fourier Transforms of Test Functions	153
1.1. Fourier Transforms of Functions in K	153
1.2. The Space Z	155
1.3. The Case of Several Variables	157
1.4. Functionals on Z	158
1.5. Analytic Functionals	160
1.6. Fourier Transforms of Functions in S	165
2. Fourier Transforms of Generalized Functions. A Single Variable	166
2.1. Definition	166
2.2. Examples	168
2.3. Fourier Transforms of x_+^λ , x_-^λ , $ x ^\lambda$, and $ x ^\lambda \operatorname{sgn} x$.	170
2.4. Fourier Transforms of $x_+^\lambda \ln x_+$ and Similar Generalized Functions	174
2.5. Fourier Transform of the Generalized Function $(ax^2 + bx + c)_+^\lambda$	182
2.6. Fourier Transforms of Analytic Functionals	188
3. Fourier Transforms of Generalized Functions. Several Variables	190
3.1. Definitions	190
3.2. Fourier Transform of the Direct Product	191
3.3. Fourier Transform of r^λ	192
3.4. Fourier Transform of Generalized Function with Bounded Support	196
3.5. The Fourier Transform as the Limit of a Sequence of Functions	200
4. Fourier Transforms and Differential Equations	200
4.1. Introductory Remarks	200
4.2. The Iterated Laplace Equation $\Delta^m u = f$	201

4.3. The Wave Equation in Space of Odd Dimension	202
4.4. The Relation between the Elementary Solution of an Equation and the Corresponding Cauchy Problem	204
4.5. Classical Operational Calculus	206
Chapter III	
Particular Types of Generalized Functions	209
1. Generalized Functions Concentrated on Smooth Manifolds of Lower Dimension	209
1.1. Introductory Remarks on Differential Forms	214
1.2. The Form ω	220
1.3. The Generalized Function $\delta(P)$	222
1.4. Example: Derivation of Green's Theorem	226
1.5. The Differential Forms $\omega_k(\varphi)$ and the Generalized Functions $\delta^{(k)}(P)$	228
1.6. Recurrence Relations for the $\delta^{(k)}(P)$	232
1.7. Recurrence Relations for the $\delta^{(k)}(aP)$	236
1.8. Multiplet Layers	237
1.9. The Generalized Function $\delta(P_1, \dots, P_k)$ and Its Derivatives	239
2. Generalized Functions Associated with Quadratic Forms	247
2.1. Definition of $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$	247
2.2. The Generalized Function P_+^λ	253
2.3. The Generalized Function \mathcal{P}^λ Associated with a Quadratic Form with Complex Coefficients	269
2.4. The Generalized Functions $(P + i0)^\lambda$ and $(P - i0)^\lambda$	274
2.5. Elementary Solutions of Linear Differential Equations .	279
2.6. Fourier Transforms of $(P + i0)^\lambda$ and $(P - i0)^\lambda$	283
2.7. Generalized Functions Associated with Bessel Functions	285
2.8. Fourier Transforms of $(c^2 + P + i0)^\lambda$ and $(c^2 + P - i0)^\lambda$	287
2.9. Fourier Transforms of $(c^2 + P)_+^\lambda$ and $(c^2 + P)_-^\lambda$	290
2.10. Fourier Transforms of $(c^2 + P)_+^\lambda/\Gamma(\lambda + 1)$ and $(c^2 + P)_-^\lambda/\Gamma(\lambda + 1)$ for Integral λ	291
3. Homogeneous Functions	295
3.1. Introduction	295
3.2. Positive Homogeneous Functions of Several Independent Variables	297

3.3. Generalized Homogeneous Functions of Degree $-n$	303
3.4. Generalized Homogeneous Functions of Degree $-n - m$	309
3.5. Generalized Functions of the Form $r^{\lambda}f$, where f Is a Generalized Function on the Unit Sphere	311
 4. Arbitrary Functions Raised to the Power λ	313
4.1. Reducible Singular Points	313
4.2. The Generalized Function G^{λ} when $G = 0$ Consists Entirely of First-Order Points	315
4.3. The Generalized Function G^{λ} when $G = 0$ Has No Points of Order Higher Than Two	318
4.4. The Generalized Function G^{λ} in General	323
4.5. Integrals of an Infinitely Differentiable Function over a Surface Given by $G = c$	326
 Summary of Fundamental Definitions and Equations of Volume I .	330
Table of Fourier Transforms	359
Appendix A. Proof of the Completeness of the Generalized-Function Space	368
Appendix B. Generalized Functions of Complex Variables	370
B1. Generalized Functions of a Single Complex Variable	371
B1.1. The Variables z and \bar{z}	371
B1.2. Homogeneous Functions of a Complex Variable	372
B1.3. The Homogeneous Generalized Functions $z^{\lambda}\bar{z}^{\mu}$	373
B1.4. The Generalized Functions z^{-k-1} and Its Derivatives . .	377
B1.5. Associated Homogeneous Functions	378
B1.6. Uniqueness Theorem for Homogeneous Generalized Functions	379
B1.7. Fourier Transforms of Test Functions and of Generalized Functions	381
B1.8. The Generalized Functions $f^{\lambda}(z) f^{\mu}(z)$, Where $f(z)$ is a Meromorphic Function	385
B2. Generalized Functions of m Complex Variables	387
B2.1. The Generalized Functions $\delta(P)$ and $\delta^{(k,l)}(P)$	387
B2.2. The Generalized Functions $G^{\lambda}\bar{G}^{\mu}$	390
B2.3. Homogeneous Generalized Functions	391
B2.4. Associated Homogeneous Functions	393
B2.5. The Residue of a Homogeneous Function	394
B2.6. Homogeneous Generalized Functions of Degree $(-m, -m)$	396

B2.7. The Generalized Function $P^\lambda P^\mu$, Where P Is a Nondegenerate Quadratic Form	398
B2.8. Elementary Solutions of Linear Differential Equations in the Complex Domain	404
B2.9. The Generalized Function $G^\lambda \bar{G}^\mu$ (General Case)	406
B2.10 Generalized Functions Corresponding to Meromorphic Functions of m Complex Variables	411
Notes and References to the Literature	413
Bibliography	416
Index	419
Index of Particular Generalized Functions	422

