CONTENTS

1 Introduction 1

1.1 The Problem to be Solved 1
1.2 Numerical Approximation of the Solution 7
1.3 An Illustration—the Euler Method 10
1.3.1 Error estimates 13
1.3.2 Comparison of error estimates with actual errors 15
1.3.3 Stability 16
1.3.4 Round-off errors 18
1.3.5 The perturbation due to the numerical approximation 21
Problems 23

2 Higher Order One-Step Methods 25

2.1 The Taylor's Series Method 25
2.2 Richardson Extrapolation to $h = 0$ 26

2.3 Second Order Runge-Kutta Methods 27
2.4 Explicit Runge-Kutta Methods 31
2.4.1 The classical Runge-Kutta method 35
2.4.2 The Ralston Runge-Kutta method 36
2.4.3 Butcher's results on the attainable order of Runge-Kutta methods 36
2.5 Implicit Runge-Kutta Methods 37
2.5.1 Practical considerations of implicit Runge-Kutta methods 39
2.6 Convergence and Stability 40
2.6.1 Stability regions for explicit Runge-Kutta methods 40
2.6.2 Stability regions for implicit Runge-Kutta methods 42
Problems 43

3 Systems of Equations and Equations of Order Greater than One 45

3.1 Application of One-Step Techniques to Systems of Equations 46
3.2 Reduction of a Higher Order Equation to a System of First Order Equations 47
3.3 Direct Methods for Higher Order Equations 47
3.3.1 Taylor's series methods 48
3.3.2 Runge-Kutta methods 48
Problems 50

4 Convergence, Error Bounds, and Error Estimates for One-Step Methods 52

4.1 Vector and Matrix Norms 53
4.2 Existence and the Lipschitz Condition 54
4.3 Convergence and Stability 55
4.4 Error Bounds and Order of Convergence 59
4.5 Asymptotic Error Estimates 61
4.5.1 The perturbation due to the numerical approximation 64

4.6 General Application of Error Bound and Estimate Theorems 66
4.6.1 Taylor's series methods 67
4.6.2 Runge-Kutta methods 67
4.6.3 The need for continuous derivatives 68
4.7 Variable Step Size 68
Problems 70

5 The Choice of Step Size and Order **72**

5.1 The Choice of Order 73
5.2 Choice of Step Size 76
5.3 The Practical Control of Errors 79
5.4 Estimation of the Local Truncation Error 81
5.4.1 Step doubling 81
5.4.2 The Runge-Kutta-Merson method 85
Problems 86

6 Extrapolation Methods **87**

6.1 Polynomial Extrapolation 87
6.1.1 Example of polynomial extrapolation 89
6.1.2 Round-off effects 89
6.1.3 Stability 89
6.1.4 Higher order methods 92
6.2 Rational Function Extrapolation 93
Problems 100

7 Multivalue or Multistep Methods—Introduction **102**

7.1 Multivalue Methods 103
7.2 Explicit Multistep Methods—the Adams-Bashforth Method 104

7.2.1 Generating functions for the coefficients 108
7.2.2 Two other techniques for deriving the Adams-Bashforth methods 109
7.2.3 Truncation error in the Adams-Bashforth methods 110
7.3 Implicit Multistep Methods—the Adams-Moulton Method 111
7.4 Predictor-Corrector Methods 114
Problems 115

8 General Multistep Methods, Order, and Stability 116

8.1 The Order of a Multistep Method 117
8.1.1 Determination of α if β is given and vice versa 119
8.1.2 The principal root of a method 120
8.2 Milne's Method 121
8.2.1 Stability of Milne's method for $y' = \lambda y$ 122
8.3 Stability of General Multistep Methods 124
8.3.1 Absolute stability 126
8.4 The Class of Three-Step Methods of Order Four 132
Problems 134

9 Multivalue Methods 136

9.1 Behavior of the Error 137
9.1.1 Stability of predictor-corrector methods 138
9.2 Equivalent Methods 142
9.2.1 Factors affecting the choice of representation 143
9.2.2 Adams' methods in the backward difference representation 147
9.2.3 The Nordsieck form of Adams' method 148
9.2.4 Modified multistep methods 150
9.2.5 Higher order equations 151
9.3 Automatic Control of Step Size and Order 155
Problems 167

10 Existence, Convergence, and Error Estimates for Multivalue Methods **169**

10.1 Convergence and Stability 172
10.1.1 Stability 174
10.1.2 Order 180
10.1.3 Consistency and convergence 187
10.2 The Maximum Order of a Stable Multistep Method 194
10.3 Existence of Stable Multivalue Methods 198
10.4 Improved Order for Normal Form Multivalue Methods 200
10.5 Asymptotic Behavior of the Error 204
Problems 207

11 Special Methods for Special Problems **209**

11.1 Stiff Equations 209
11.1.1 Multistep methods 212
11.1.2 A-stable methods 220
11.1.3 Methods based on a knowledge of $\delta f/\delta y$ 222
11.2 Algebraic and Singular Equations 223
11.3 Parameter Estimation 227
Problems 229

12 Choosing a Method **231**

12.1 Effect of Future Developments 235

Bibliography **237**

Index **251**