Contents

Preface *ix*
General Notation *xi*
Contents of Volume 2 *xiii*

1 Stochastic Processes

1. The Kolmogorov construction of a stochastic process 1
2. Separable and continuous processes 6
3. Martingales and stopping times 9
Problems 15

2 Markov Processes

1. Construction of Markov processes 18
2. The Feller and the strong Markov properties 23
3. Time-homogeneous Markov processes 30
Problems 31

3 Brownian Motion

1. Existence of continuous Brownian motion 36
2. Nondifferentiability of Brownian motion 39
3. Limit theorems 40
4. Brownian motion after a stopping time 44
5. Martingales and Brownian motion 46
6. Brownian motion in n dimensions 50
Problems 53

4 The Stochastic Integral

1. Approximation of functions by step functions 55
2. Definition of the stochastic integral 59
3. The indefinite integral 67
4. Stochastic integrals with stopping time 72
5. Itô's formula 78
6. Applications of Itô's formula 85
7. Stochastic integrals and differentials in n dimensions 89

Problems 93

5 Stochastic Differential Equations

1. Existence and uniqueness 98
2. Stronger uniqueness and existence theorems 102
3. The solution of a stochastic differential system as a Markov process 108
4. Diffusion processes 114
5. Equations depending on a parameter 117
6. The Kolmogorov equation 123

Problems 125

6 Elliptic and Parabolic Partial Differential Equations and Their Relations to Stochastic Differential Equations

1. Square root of a nonnegative definite matrix 128
2. The maximum principle for elliptic equations 132
3. The maximum principle for parabolic equations 134
4. The Cauchy problem and fundamental solutions for parabolic equations 139
5. Stochastic representation of solutions of partial differential equations 144

Problems 150

7 The Cameron–Martin–Girsanov Theorem

1. A class of absolutely continuous probabilities 152
2. Transformation of Brownian motion 156
3. Girsanov's formula 164

Problems 169

8 Asymptotic Estimates for Solutions

1. Unboundedness of solutions 172
2. Auxiliary estimates 174
3. Asymptotic estimates 180
4. Applications of the asymptotic estimates 185
5. The one-dimensional case 188
6. Counterexample 191
Problems 193

9 Recurrent and Transient Solutions

1. Transient solutions 196
2. Recurrent solutions 200
3. Rate of wandering out to infinity 203
4. Obstacles 207
5. Transient solutions for degenerate diffusion 213
6. Recurrent solutions for degenerate diffusion 217
7. The one-dimensional case 219
Problems 222

Bibliographical Remarks 224
References 226
Index 1