

Contents

Preface	vii
Introduction	1
I. Dirichlet's Principle and the Boundary Value Problem of Potential Theory	5
1. Dirichlet's Principle.....	5
Definitions.....	5
Original statement of Dirichlet's Principle.....	6
General objection: A variational problem need not be solvable.....	6
Minimizing sequences.....	8
Explicit expression for Dirichlet's integral over a circle. Specific objection to Dirichlet's Principle.....	9
Correct formulation of Dirichlet's Principle.....	10
2. Semicontinuity of Dirichlet's integral. Dirichlet's Principle for circular disk.....	11
3. Dirichlet's integral and quadratic functionals.....	13
4. Further preparation.....	16
Convergence of a sequence of harmonic functions.....	16
Oscillation of functions appraised by Dirichlet's integral.....	18
Invariance of Dirichlet's integral under conformal mapping. Applications.....	20
Dirichlet's Principle for a circle with partly free boundary.....	21
5. Proof of Dirichlet's Principle for general domains.....	23
Direct methods in the calculus of variations.....	23
Construction of the harmonic function u by a "smoothing process".....	24
Proof that $D[u] = d$	28
Proof that u attains prescribed boundary values.....	28
Generalizations.....	30
6. Alternative proof of Dirichlet's Principle.....	31
Fundamental integral inequality.....	31
Solution of variational problem I.....	32
7. Conformal mapping of simply and doubly connected domains.....	38
8. Dirichlet's Principle for free boundary values. Natural boundary conditions.....	40

II. Conformal Mapping on Parallel-Slit Domains.....	45
1. Introduction.....	45
Classes of normal domains. Parallel-slit domains.....	45
Variational problem: Motivation and formulation.....	48
2. Solution of variational problem II.....	51
Construction of the function u	51
Continuous dependence of the solution on the domain.....	54
3. Conformal mapping of plane domains on slit domains.....	55
Mapping of k -fold connected domains.....	56
Mapping on slit domains for domains G of infinite connectivity.....	58
Half-plane slit domains. Moduli.....	61
Boundary mapping.....	62
4. Riemann domains.....	64
Introduction.....	64
The "sewing theorem".....	69
5. General Riemann domains. Uniformization.....	75
6. Riemann domains defined by non-overlapping cells.....	78
7. Conformal mapping of domains not of genus zero.....	80
Introduction.....	80
Description of slit domains not of genus zero.....	80
The mapping theorem.....	85
Remarks. Half-plane slit domains.....	92
 III. Plateau's Problem.....	 95
1. Introduction.....	95
2. Formulation and solution of basic variational problems.....	101
Notations.....	101
Fundamental lemma. Solution of minimum problem.....	101
Remarks. Semicontinuity.....	104
3. Proof by conformal mapping that solution is a minimal surface.....	105
4. First variation of Dirichlet's integral.....	107
Variation in general space of admissible functions.....	107
First variation in space of harmonic vectors.....	110
Proof that stationary vectors represent minimal surfaces.....	112
5. Additional remarks.....	115
Biunique correspondence of boundary points.....	115
Relative minima.....	115
Proof that solution of variational problem solves problem of least area.....	116
Role of conformal mapping in solution of Plateau's problem.....	117
6. Unsolved problems.....	118
Analytic extension of minimal surfaces.....	118
Uniqueness. Boundaries spanning infinitely many minimal surfaces.....	119
Branch points of minimal surfaces.....	122

III. Plateau's Problem—Continued

7. First variation and method of descent.....	123
8. Dependence of area on boundary.....	126
Continuity theorem for absolute minima.....	126
Lengths of images of concentric circles.....	127
Isoperimetric inequality for minimal surfaces.....	129
Continuous variation of area of minimal surfaces.....	131
Continuous variation of area of harmonic surfaces.....	134

IV. The General Problem of Douglas..... 141

1. Introduction.....	141
2. Solution of variational problem for k -fold connected domains....	144
Formulation of problem.....	144
Condition of cohesion.....	145
Solution of variational problem for k -fold connected domains	
G and parameter domains bounded by circles.....	146
Solution of variational problem for other classes of normal do-	
mains.....	149
3. Further discussion of solution.....	149
Douglas' sufficient condition.....	149
Lemma 4.1 and proof of theorem 4.2.....	151
Lemma 4.2 and proof of theorem 4.1.....	153
Remarks and examples.....	158
4. Generalization to higher topological structure.....	160
Existence of solution.....	160
Proof for topological type of Moebius strip.....	161
Other types of parameter domains.....	164
Identification of solutions as minimal surfaces. Properties of	
solution.....	165

V. Conformal Mapping of Multiply Connected Domains..... 167

1. Introduction.....	167
Objective.....	167
First variation.....	168
2. Conformal mapping on circular domains.....	169
Statement of theorem.....	169
Statement and discussion of variational conditions.....	169
Proof of variational conditions.....	171
Proof that $\Phi(w) = 0$	175
3. Mapping theorems for a general class of normal domains.....	178
Formulation of theorem.....	178
Variational conditions.....	179
Proof that $\Phi(w) = 0$	180
4. Conformal mapping on Riemann surfaces bounded by unit circles... 183	
Formulation of theorem.....	183

V. Conformal Mapping of Multiply Connected Domains—*Continued*

Variational conditions. Variation of branchpoints.....	184
Proof that $\Phi(w) = 0$	186
5. Uniqueness theorems.....	187
Method of uniqueness proof.....	187
Uniqueness for Riemann surfaces with branch points.....	188
Uniqueness for classes \mathfrak{R} of plane domains.....	188
Uniqueness for other classes of domains.....	190
6. Supplementary remarks.....	191
First continuity theorem in conformal mapping.....	191
Second continuity theorem. Extension of previous mapping theorems.....	191
Further observations on conformal mapping.....	192
7. Existence of solution for variational problem in two dimensions....	192
Proof using conformal mapping of doubly connected domains....	192
Alternative proof. Supplementary remarks.....	197

VI. Minimal Surfaces with Free Boundaries and Unstable Minimal Surfaces.....

1. Introduction.....	199
Free boundary problems.....	199
Unstable minimal surfaces.....	200
2. Free boundaries. Preparations.....	201
General remarks.....	201
A theorem on boundary values.....	202
3. Minimal surfaces with partly free boundaries.....	206
Only one arc fixed.....	206
Remarks on Schwarz' chains.....	208
Doubly connected minimal surfaces with one free boundary....	209
Multiply connected minimal surfaces with free boundaries....	211
4. Minimal surfaces spanning closed manifolds.....	213
Introduction.....	213
Existence proof.....	214
5. Properties of the free boundary. Transversality.....	218
Plane boundary surface. Reflection.....	218
Surface of least area whose free boundary is not a continuous curve.....	220
Transversality.....	222
6. Unstable minimal surfaces with prescribed polygonal boundaries...	223
Unstable stationary points for functions of N variables.....	223
A modified variational problem.....	226
Proof that stationary values of $d(U)$ are stationary values for $D[\mathfrak{r}]$	232
Generalization.....	233
Remarks on a variant of the problem and on second variation...	235
7. Unstable minimal surfaces in rectifiable contours.....	236

VI. Minimal Surfaces with Free Boundaries and Unstable Minimal Surfaces—*Continued*

Preparations. Main theorem.....	236
Remarks and generalizations.....	240
8. Continuity of Dirichlet's integral under transformation of \mathfrak{r} -space..	241
Bibliography, Chapters I to VI.....	245
Appendix. Some Recent Developments in the Theory of Conformal Mapping. By M. SCHIFFER.....	249
1. Green's function and boundary value problems.....	249
Canonical conformal mappings.....	253
Boundary value problems of second type and Neumann's function.....	259
2. Dirichlet integrals for harmonic functions.....	266
Formal remarks.....	266
The kernels K and L	268
Inequalities.....	273
Conformal transformations.....	275
An application to the theory of univalent functions... ..	276
Discontinuities of the kernels.....	277
An eigenvalue problem.....	278
Kernel functions for the class \mathfrak{F}_0	281
Comparison theory.....	283
An extremum problem in conformal mapping.....	289
Mapping onto a circular domain.....	290
Orthornormal systems.....	291
3. Variation of the Green's function.....	292
Hadamard's variation formula.....	292
Interior variations.....	298
Application to the coefficient problem for univalent functions.....	300
Boundary variations.....	306
Lavrentieff's method.....	310
Method of extremal length.....	313
Concluding remarks.....	317
Bibliography to Appendix.....	319
Index.....	325