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A student to whom the subject is new should peruse Chapter I., not dwelling upon it, but returning to it as he finds occasion ; and he may afterwards in the first instance confine himself to Chapters II., III., IV., XII. and XIII.

